MINIMAL DISKS BOUNDED BY THREE STRAIGHT LINES IN EUCLIDEAN SPACE AND TRINOIDS IN HYPERBOLIC SPACE

BENOÎT DANIEL

ABSTRACT. Following Riemann's idea, we prove the existence of a minimal disk in Euclidean space bounded by three lines in generic position and with three helicoidal ends of angles less than π. In the case of general angles, we prove that there exist at most four such minimal disks, we give a sufficient condition of existence in terms of a system of three equations of degree 2, and we give explicit formulas for the Weierstrass data in terms of hypergeometric functions. Finally, we construct constant-mean-curvature-one trinoids in hyperbolic space by the method of the conjugate cousin immersion.

1. INTRODUCTION

In this paper we investigate minimal disks in Euclidean space $\mathbb{R}^3$ bounded by three straight lines in generic position (*i.e.* the lines do not intersect one another and do not lie in parallel planes, in particular they are not pairwise parallel). This problem was investigated by Riemann in his posthumous memoir [Rie68]. He actually introduced the spinor representation, the Gauss map and the Hopf differential of minimal surfaces in Euclidean space. He investigated the case of minimal surfaces bounded by a contour composed of pieces of straight lines (possibly going to infinity). He studied more precisely the cases where the contour is composed of 2, 3 or 4 lines.

However, his study was not complete and sometimes not precise; in particular, he did not deal with questions of orientations. The first aim of this paper is to complete Riemann's study of minimal surfaces bounded by three straight lines. More precisely we will investigate minimal immersions x from $\Sigma = \{z \in \mathbb{C} | \operatorname{Im} z \geqslant 0\} \setminus \{0, 1\}$ into $\mathbb{R}^3$ mapping $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ onto three lines (in generic position) and having helicoidal ends (in the sense explained in section 2.2) at 0, 1 and ∞. The method is to study the Weierstrass data of x and to use the spinor representation.

We first prove that there exist at most four minimal immersions bounded by three given lines (in generic position) with helicoidal ends of given parameters (theorem 21). To do this, we use a result by Riemann: he proved that the spinor data satisfy a differential equation involving the Schwarzian derivative of the Gauss map. This is a second order equation with five regular singularities. Studying the behaviour of the Schwarzian derivative at these singular points, we prove that the Schwarzian derivative only depends on two parameters that are related by two polynomial equations of degree 2, and thus that there are at most four possiblities for the Schwarzian derivative of the Gauss map.

We next study the explicit immersion given by Riemann in his memoir [Rie68]: he introduced spinors in terms of hypergeometric functions, but he did not check that they actually gave a minimal immersion bounded by given lines. We establish this fact in proposition 31. More precisely, given three lines in generic position, denoting by A, B, C the distances between the lines, and $\pi\alpha$, $\pi\beta$, $\pi\gamma$ the angles of the ends (with signs as explained in section 2.2), we prove

1991 *Mathematics Subject Classification.* Primary: 53A10. Secondary: 53C42, 53A35, 30F45.

Key words and phrases. Minimal surfaces, Euclidean space, Weierstrass representation, Bryant surfaces, hyperbolic space, constant mean curvature.

that to each real solution (p,q,r) (up to a sign) of the system

$$\begin{cases} p^2-\alpha^2(p+q+r)^2 &= \varepsilon\frac{A\alpha}{2\pi} \\ q^2-\beta^2(p+q+r)^2 &= \varepsilon\frac{B\beta}{2\pi} \\ r^2-\gamma^2(p+q+r)^2 &= \varepsilon\frac{C\gamma}{2\pi} \end{cases}$$

where $\varepsilon \in \{1,-1\}$ (depending on the geometric configuration of the lines and on the angles) corresponds a minimal immersion (with possibly a singular point when the Hopf differential has a double zero, which is a non-generic situation) bounded by these lines and with helicoidal ends of parameters (A,α), (B,β), (C,γ) (theorem 33). In particular we prove the existence of at least one minimal immersion in the case where the angles are less than π (corollary 34).

However we do not know if we obtain all the solutions in this way.

Figure 1 is a picture of a minimal surface bounded by three lines with helicoidal ends, drawn with the software "Evolver".

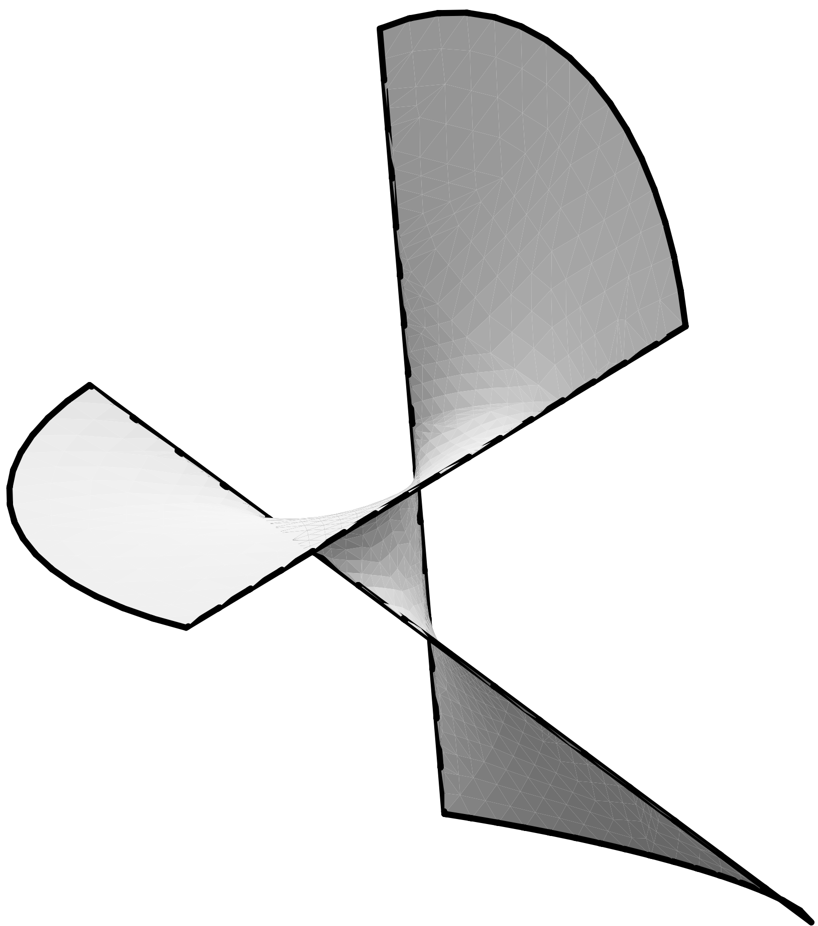

FIGURE 1. A minimal surface with three helicoidal ends.

In the last part of this paper, we construct constant-mean-curvature-one (CMC-1) trinoids in hyperbolic space $\mathbb{H}^3$ applying the conjugate cousin method to minimal disks bounded by three straight lines in $\mathbb{R}^3$. CMC-1 surfaces in $\mathbb{H}^3$ are called Bryant surfaces. Bryant proved in [Bry87] that they are closely related to minimal surfaces in $\mathbb{R}^3$ (see also [UY93] and [Ros02]); in particular there exists a representation in terms of holomorphic data analogous to Weierstrass representation.

Irreducible trinoids in $\mathbb{H}^3$ were first classified by Umehara and Yamada in [UY00], and then by Bobenko, Pavlyukevich and Springborn in [BPS02], using different techniques: their method

has some similarities with that used in this paper to find minimal surfaces bounded by three lines in $\mathbb{R}^3$ (they use a spinor representation for Bryant surfaces and they obtain explicit formulas in terms of hypergeometric functions). The technique of the conjugate cousin immersion was used by Karcher in [Kar01] to construct trinoids with dihedral symmetry. Here we use this technique to construct general irreducible trinoids with a symmetry plane (actually every irreducible trinoid has a symmetry plane by the classification of [UY00]). We also prove that the asymptotic boundary points of the ends of these trinoids are distinct (except in exceptionnal cases) (theorem 49). Finally we give examples of minimal disks bounded by three lines whose conjugate cousins are invariant by some parabolic isometries.

Acknowledgements. The author is grateful to Pascal Collin for his explainations on Riemann's memoir [Rie68], and to Harold Rosenberg for submitting this problem and for discussions about this paper.

2. Preliminaries

In all this paper, we will set $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$,

$$\Sigma = \{z \in \mathbb{C} | \operatorname{Im} z \geqslant 0\} \setminus \{0, 1\}, \quad \Sigma_0 = \{z \in \Sigma | |z| < 1\},$$

$$\Sigma_1 = \{z \in \Sigma | |z-1| < 1\}, \quad \Sigma_\infty = \{z \in \Sigma | |z| > 1\}.$$

The canonical scalar product of $\mathbb{R}^3$ is denoted by $< \cdot, \cdot >$. If D is a straight line in $\mathbb{R}^3$, then $D^\perp$ denotes the set of unit vectors that are orthogonal to D. The canonical basis of $\mathbb{R}^3$ will be denoted by $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$:

$$\vec{e}_1 = (1, 0, 0), \quad \vec{e}_2 = (0, 1, 0), \quad \vec{e}_3 = (0, 0, 1).$$

We define the logarithm and non-integer powers on Σ in the following way. Let $\kappa \in \mathbb{R}$. If $z = \rho e^{i\theta} \in \Sigma$ with $\rho > 0$ and $\theta \in [0, \pi]$, then $\ln z = \ln \rho + i\theta$ and $z^\kappa = \rho^\kappa e^{i\kappa\theta}$. If $z \in \Sigma$ and $z - 1 = \rho e^{i\theta}$ with $\rho > 0$ and $\theta \in [0, \pi]$, then $(z-1)^\kappa = \rho^\kappa e^{i\kappa\theta}$ and $(1-z)^\kappa = \rho^\kappa e^{i\kappa(\theta - \pi)} = e^{-i\pi\kappa}(z-1)^\kappa$ (this convention is chosen in order that $(1-z)^\kappa$ is real when z is real and less than 1).

Finally, $\mathcal{D}$ denotes the set of the triples of straight lines in $\mathbb{R}^3$ that are neither pairwise concurrent, neither pairwise parallel, nor lying in parallel planes, modulo direct isometries of $\mathbb{R}^3$.

2.1. Weierstrass representation.

In this section we recall basic facts about Weierstrass representation and we introduce some notations.

Let $\mathcal{S}$ be a Riemann surface with boundary, and $x = (x_1, x_2, x_3) : \mathcal{S} \to \mathbb{R}^3$ a conformal minimal immersion. Then we have

$$x(z) = x(z_0) + \operatorname{Re} \int_{z_0}^{z} \big((1 - g^2), i(1 + g^2), 2g\big)\, \omega$$

where z_0 is a fixed point in $\mathcal{S}$ and (g, ω) the Weierstrass data of x: g is a meromorphic function on $\mathcal{S}$ and ω a holomorphic 1-form on $\mathcal{S}$. The poles of g are the zeros of ω, and z is a pole of g of order k if and only if z is a zero of ω of order $2k$. Conversely, if g and ω satisfy this condition, then they define a minimal immersion.

We define $X = (X_1, X_2, X_3) : \mathcal{S} \to \mathbb{C}^3$ by

$$X(z) = x(z_0) + \int_{z_0}^{z} \big((1 - g^2), i(1 + g^2), 2g\big)\, \omega.$$

We have

$$\mathrm{d}x_1 + i\mathrm{d}x_2 = \bar{\omega} - g^2\omega, \quad \mathrm{d}x_3 = \operatorname{Re}(2g\omega).$$

The Gauss map of x is

$$N = \left(\frac{2g}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right).$$

The orientations induced on $x(\mathcal{S})$ by the Gauss map N and the immersion x are compatible. The function g is the composition of the Gauss map and the stereographic projection with respect to the north pole of the sphere. If D is a line parallel to the vector $(\cos(\pi\alpha), \sin(\pi\alpha), 0)$ for $\alpha \in \mathbb{R}$, then the circle $D^\perp$ corresponds to $g \in ie^{i\pi\alpha}\bar{\mathbb{R}}$.

If f is a meromorphic function on an open set $U \subset \mathcal{S}$, we define its Schwarzian derivative with respect to a local conformal coordinate z by

$$\mathrm{S}_z f = \left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right)\mathrm{d}z^2.$$

If ζ is another local conformal coordinate, then $\mathrm{S}_z f = \mathrm{S}_\zeta f + \mathrm{S}_z \zeta$. If f is regular at a point z_1, then $\mathrm{S}_z f$ is holomorphic at z_1; if f has a branch point of order $j-1$ at z_1 with $j \geqslant 2$, then $\mathrm{S}_z f$ has a pole of order 2 at z_1, and its coefficient of order -2 is equal to $\frac{1-j^2}{2}$.

The Hopf differential is the holomorphic 2-form on $\mathcal{S}$ defined by

$$Q = \omega \mathrm{d}g = \frac{1}{2}\mathrm{d}X_3 \frac{\mathrm{d}g}{g}.$$

The forms Q and $\mathrm{S}_z g$ are invariant by a direct isometry of $\mathbb{R}^3$. The first and second fondamental forms of the surface given by

$$\mathrm{I} = (1+|g|^2)^2|\omega|^2, \quad \mathrm{II} = -2\,\mathrm{Re}\, Q.$$

Proposition 1. *Let $z \in \mathcal{S}$ and $k \in \mathbb{N}^*$. Then z is a zero of Q of order k if and only if z is a branch point of g of order k. This happens if and only if one of the following conditions holds:*

- *z is a zero of g of order $k+1$,*
- *z is a pole of g of order $k+1$,*
- *z is a zero of $\frac{\mathrm{d}g}{g}$ of order k.*

Proof. If z is a zero of g of order $d \in \mathbb{N}^*$, then it is not a zero of ω and it is a simple pole of $\frac{\mathrm{d}g}{g}$. Consequently it is a zero of Q of order k if and only if $k = d-1$.

If z is a pole of g of order $d \in \mathbb{N}^*$, then it is a zero of ω of order $2d$ and it is a simple pole of $\frac{\mathrm{d}g}{g}$. Consequently it is a zero of Q of order k if and only if $k = d-1$.

If z is neither a zero nor a pole of g, then it is not a zero of ω, and consequently it is a zero of Q of order k if and only if it is a zero of $\frac{\mathrm{d}g}{g}$ of order k. □

We recall the spinor representation of a minimal surface:

$$x(z) = x(z_0) + \mathrm{Re}\int_{z_0}^{z} \left(\xi_1^2 - \xi_2^2, i(\xi_1^2+\xi_2^2), 2\xi_1\xi_2\right)$$

where ξ_2 and ξ_2 are holomorphic sections of a spin structure (see [KS96] for more information). These spinors satisfy

$$g = \frac{\xi_2}{\xi_1}, \quad \omega = \xi_1^2.$$

Two such holomorphic spinors define a minimal immersion if and only if they do not have common zeros (if they have a common zero, then the map x is not an immersion at this point). We will call (ξ_1, ξ_2) the spinor data of x.

2.2. **Helicoidal ends.** Most of the results of this section are contained in Riemann's memoir [Rie68]. Here we prove these results using modern formalism, and we give precise definitions for helicoidal ends and the signs of distances and angles.

Definition 2. *Let D_1 and D_2 be two nonparallel and nonconcurrent oriented straight lines, and let $\vec{u}_1$ and $\vec{u}_2$ be unit vectors inducing the orientations of D_1 and D_2. Then the unit vector*

$$\vec{v} = \frac{\vec{u}_1 \times (-\vec{u}_2)}{||\vec{u}_1 \times \vec{u}_2||}$$

is called the vector associated to the couple (D_1, D_2) of oriented straight lines.

Definition 3. *The signed distance of D_1 and D_2 is the number $\mathrm{D}(D_1, D_2) =< \overrightarrow{p_1p_2}, \vec{v} >$ where $p_1 \in D_1$ and $p_2 \in D_2$ (this number does not depend on the choices of p_1 and p_2, and $|\mathrm{D}(D_1, D_2)|$ is the distance between D_1 and D_2).*

Definition 4. *Let U be a neighbourhood of 0 in $\mathbb{C}$ that is symmetric with respect to the real axis (i.e. $z \in U \iff \bar{z} \in U$), $\Omega = \Sigma \cap U$, $\Omega_1 = \Omega \cap (-\infty, 0)$, $\Omega_2 = \Omega \cap (0, +\infty)$ and $x : \Omega \to \mathbb{R}^3$ be a conformal minimal immersion that is complete at 0. Let D_1 and D_2 be two nonparallel and nonconcurrent oriented straight lines, let $\vec{u}_1$ and $\vec{u}_2$ be unit vectors inducing the orientations of D_1 and D_2, and let $\vec{v}$ be the vector associated to (D_1, D_2).*

We say that the immersion x has an end (at $z = 0$) bounded by the couple of oriented lines (D_1, D_2) if

1. *the immersion x maps Ω_1 to a part of D_1 and $< x(z), \vec{u}_1 > \to +\infty$ when $z \to 0$ with z real and negative,*
2. *the immersion x maps Ω_2 to a part of D_2 and $< x(z), \vec{u}_2 > \to -\infty$ when $z \to 0$ with z real and positive.*

We say that the immersion x has a helicoidal end (at $z = 0$) bounded by the couple of oriented lines (D_1, D_2) if moreover the two following conditions are satisfied:

3. *the Gauss map of x has a limit when $z \to 0$,*
4. *the quantity $< x(z), \vec{v} >$ is bounded when $z \to 0$.*

It will follow from the proof of lemma 7 that a helicoidal end is actually asymptotic to a helicoid.

Lemma 5. *Assume that x has a helicoidal end bounded by (D_1, D_2). Let N be the Gauss map of x. Then the limit point of N at 0 is $N(0) = \vec{v}$ or $N(0) = -\vec{v}$.*

Proof. We have $N(z) \in D_1^\perp$ if $z \in \Omega_1$ and $N(z) \in D_2^\perp$ if $z \in \Omega_2$, so $N(0) \in D_1^\perp \cap D_2^\perp$. □

Lemma 6. *Assume that x has an end bounded by (D_1, D_2). Let (g, ω) be the Weierstrass data of x, and Q its Hopf differential. Then Q extends to a holomorphic 2-form on $U \setminus \{0\}$ and $\mathrm{S}_z g$ extends to a meromorphic 2-form on $U \setminus \{0\}$.*

Proof. Since a direct isometry of $\mathbb{R}^3$ does not change Q and $\mathrm{S}_z g$, we can assume that D_2 is the x_1-axis. Then x_3 is constant on Ω_2, so $x_3' = 0$ on Ω_2, and thus $X_1'(z) \in i\mathbb{R}$ for $z \in \Omega_2$. And the Gauss map N is normal to the x_1-axis on Ω_2, so $g(\Omega_2) \subset i\bar{\mathbb{R}}$, and thus $\frac{g'}{g}(z) \in \bar{\mathbb{R}}$ for $z \in \Omega_2$. Thus $\frac{Q}{\mathrm{d}z^2} = \frac{1}{2} X_3' \frac{g'}{g}$ is purely imaginary on Ω_2 (since it cannot be infinite). The same holds on Ω_1. Thus we can apply the Schwarz reflection to $\frac{Q}{\mathrm{d}z^2}$ (which is up to now defined on Ω), and we obtain a holomorphic 2-form Q defined on $U \setminus \{0\}$.

In the same way, assuming that D_2 is the x_1-axis, we have $\frac{g''}{g'}(z) \in \bar{\mathbb{R}}$ for $z \in \Omega_2$, and so $(\frac{g''}{g'})' - \frac{1}{2}(\frac{g''}{g'})^2$ is real or infinite on Ω_2. The same holds on Ω_1, and we obtain a meromorphic 2-form $\mathrm{S}_z g$ defined on $U \setminus \{0\}$ by Schwarz reflection. □

Lemma 7. *Assume that x has an end bounded by (D_1, D_2). Let (g, ω) be the Weierstrass data of x, and Q its Hopf differential. If x has a helicoidal end bounded by (D_1, D_2), then there exist $A \in \mathbb{R}^*$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ such that*

$$Q \sim i\frac{A\alpha}{2\pi} z^{-2}\mathrm{d}z^2, \quad \mathrm{S}_z g \sim \frac{1-\alpha^2}{2} z^{-2}\mathrm{d}z^2 \tag{1}$$

when $z \to 0$.

The couple (A, α) is then defined uniquely up to a sign. Moreover, if $\pi\alpha_0$ denotes the angle of $\vec{u}_1$ and $-\vec{u}_2$ with $\alpha_0 \in (0,1)$, then we have either $\alpha \in \alpha_0 + 2\mathbb{Z}$ and $A = -\mathrm{D}(D_1, D_2)$, or $-\alpha \in \alpha_0 + 2\mathbb{Z}$ and $A = \mathrm{D}(D_1, D_2)$.

We say that x has a helicoidal end of parameters (A, α), and that $\pi\alpha$ is the angle of the helicoidal end.

Proof. Since a direct isometry of $\mathbb{R}^3$ does not change Q and $\mathrm{S}_z g$, we can assume that $\vec{u}_1 = (\cos(\pi\alpha_0), \sin(\pi\alpha_0), 0)$, $\vec{u}_2 = -\vec{e}_1$, D_1 is the line $(0,0,-A) + \mathbb{R}\vec{u}_1$ and D_2 the x_1-axis. Then we have $\vec{v} = -\vec{e}_3$, and so $g(0) = 0$ or $g(0) = \infty$ by lemma 5. Moreover we have $\mathrm{D}(D_1, D_2) = -A$.

Set $h(z) = z^{-\alpha_0} g(z)$ for $z \in \Omega$. Then $h(z) \in i\bar{\mathbb{R}}$ if $z \in \Omega_1$ (since $N(z) \in D_1^\perp$) or $z \in \Omega_2$ (since $N(z) \in D_2^\perp$). Thus h extends to a meromorphic map on $U \setminus \{0\}$ by Schwarz reflection principle. Since g has a limit at 0, h has no essential singularity at 0.

Thus there exist a integer j and a nonzero real number ρ such that $g(z) \sim i\rho z^{\alpha_0 + j}$. We set $\alpha = \alpha_0 + j$. We compute that $\mathrm{S}_z g \sim \frac{1-\alpha^2}{2} z^{-2}\mathrm{d}z^2$.

Let $h_1(z) = X_3(z) - i\frac{A}{\pi}\ln z$ for $z \in \Omega$, with X_3 as in section 2.1. Then, since $x_3 = \operatorname{Re} X_3$, we have $h_1(z) \in i\mathbb{R}$ if $z \in \Omega_1$ or $z \in \Omega_2$. Thus h_1 extends to a meromorphic map on $U \setminus \{0\}$ by Schwarz reflection principle. Moreover, $x_3(z) = -< x(z), \vec{v} >$ is bounded in the neighbourhood of 0 by condition 4 in definition 4, so h_1 is holomorphic at 0.

Hence we have $\mathrm{d}X_3 \sim i\frac{A}{\pi} z^{-1}\mathrm{d}z$, and so $Q = \frac{1}{2}\mathrm{d}X_3 \frac{\mathrm{d}g}{g} \sim i\frac{A\alpha}{2\pi} z^{-2}\mathrm{d}z^2$.

We now prove that the integer $j = \alpha - \alpha_0$ is even. We have

$$\mathrm{d}(x_1 + ix_2) = \bar{\omega} - g^2\omega = \frac{\overline{\mathrm{d}X_3}}{2\bar{g}} - \frac{g\mathrm{d}X_3}{2} \sim \frac{A}{2\pi}(\rho^{-1}\bar{z}^{-1-\alpha}\overline{\mathrm{d}z} + \rho z^{-1+\alpha}\mathrm{d}z).$$

We set $z = t + i\tau$ with t and τ real. We have $< x, \vec{u}_1 >= \operatorname{Re}((x_1 + ix_2)e^{-i\pi\alpha_0})$, so for $t < 0$ we have

$$\frac{\partial}{\partial t} < x, \vec{u}_1 > \sim -\frac{A}{2\pi}\cos(\pi(\alpha - \alpha_0))(\rho^{-1}|t|^{-1-\alpha} + \rho|t|^{-1+\alpha}).$$

Thus condition 1 in definition 4 implies that $-A\rho\cos(\pi(\alpha - \alpha_0)) > 0$. And we have $< x, -\vec{u}_2 >= x_1 = \operatorname{Re}(x_1 + ix_2)$, so for $t > 0$ we have

$$\frac{\partial}{\partial t} < x, -\vec{u}_2 > \sim \frac{A}{2\pi}(\rho^{-1}t^{-1-\alpha} + \rho t^{-1+\alpha}).$$

Thus condition 2 in definition 4 implies that $A\rho < 0$. We conclude that $\cos(\pi(\alpha - \alpha_0)) > 0$, that is $\alpha - \alpha_0 \in 2\mathbb{Z}$.

Finally, it is clear that (1) defines (A, α) uniquely up to a sign. □

Let x be an immersion having a helicoidal end of parameters (A, α) bounded by two lines D_1 and D_2 that are as in the proof of this lemma. Without loss of generality we can assume that $\alpha \in \alpha_0 + 2\mathbb{Z}$, and thus $A = -\mathrm{D}(D_1, D_2)$. Then $\alpha > 0$ means that the Gauss map at 0 points down and that we turn in the clockwise direction when we go from D_1 to D_2 on the minimal surface, and $\alpha < 0$ means that the Gauss map at 0 points up and that we turn in the counter-clockwise direction when we go from D_1 to D_2 on the minimal surface. On the other hand, $A > 0$ means that D_1 lies below D_2, and $A < 0$ means that D_1 lies above D_2. Thus, $A\alpha > 0$ means that we go down when we turn in the counter-clockwise direction on the minimal

surface, and $A\alpha < 0$ means that we go up when we turn in the counter-clockwise direction on the minimal surface. This last fact remains true if $-\alpha \in \alpha_0 + 2\mathbb{Z}$. Hence we say that x has a left-helicoidal end (respectively a right-helicoidal end) if $A\alpha > 0$ (respectively $A\alpha < 0$) (see figures 2, 3, 4 and 5).

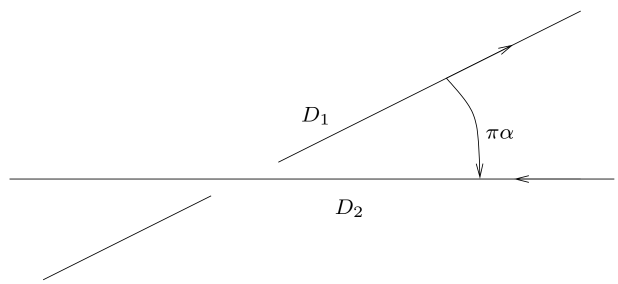

Figure 2. $A > 0$ and $\alpha > 0$ (left-helicoidal end).

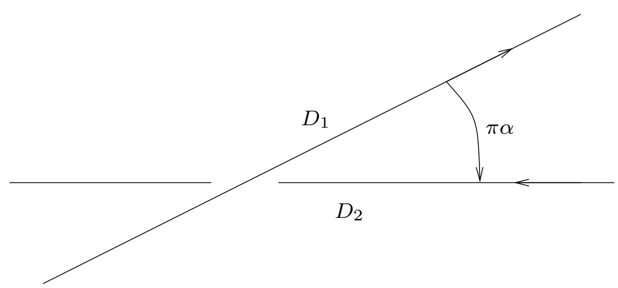

Figure 3. $A < 0$ and $\alpha > 0$ (right-helicoidal end).

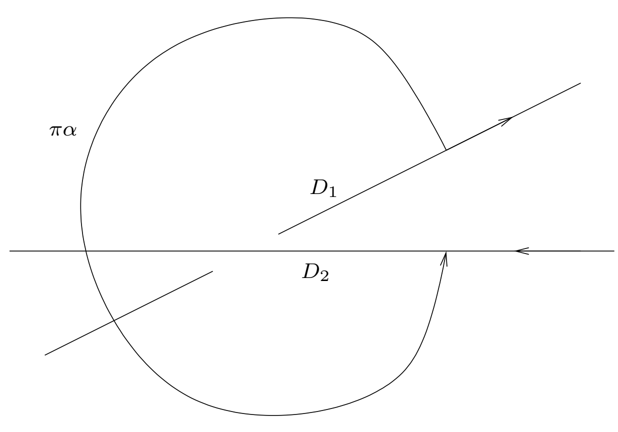

Figure 4. $A > 0$ and $\alpha < 0$ (right-helicoidal end).

Remark 8. *This definition and these lemmas extend for ends at a point* $z_1 \in \mathbb{R}$.

They also extend for end at ∞ *using the change of parameters* $\zeta = -z^{-1}$, *which maps* $\{\operatorname{Im} z > 0\}$ *onto itself. We get*

$$Q \sim i\frac{A\alpha}{2\pi} z^{-2}\mathrm{d}z^2, \quad \mathrm{S}_z g \sim \frac{1-\alpha^2}{2} z^{-2}\mathrm{d}z^2$$

when $z \to \infty$ *(because* $\mathrm{S}_z g = \mathrm{S}_\zeta g + \mathrm{S}_z \zeta$ *and* $\mathrm{S}_z\zeta = 0$*).*

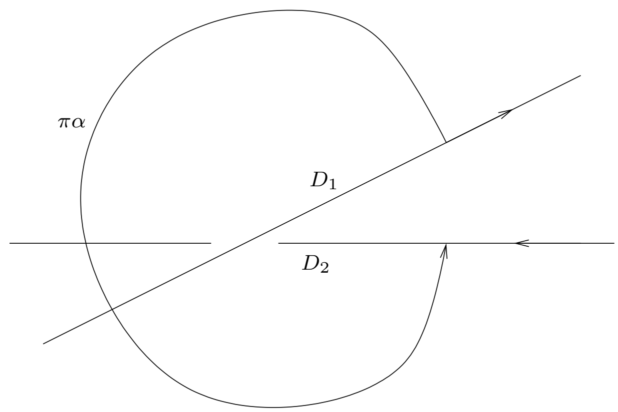

FIGURE 5. $A < 0$ and $\alpha < 0$ (left-helicoidal end).

3. Minimal surfaces bounded by three lines with helicoidal ends

In this section we will study minimal disks bounded by three lines with three helicoidal ends when the triple of lines belong to $\mathcal{D}$, which is a generic property.

3.1. **Geometric configuration.** An element of $\mathcal{D}$ has a representant that is described as follows.

Let D_1 be the horizontal line oriented by the vector $\vec{u}_1 = (\cos(\pi\alpha_0), \sin(\pi\alpha_0), 0)$ for some $\alpha_0 \in (0,1)$, D_2 the x_1-axis oriented by the vector $\vec{u}_2 = -\vec{e}_1$, and D_3 the line oriented by the vector $\vec{u}_3 = (\cos(\pi\gamma')\sin\kappa, -\sin(\pi\gamma')\sin\kappa, \cos\kappa)$ for some $\gamma' \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ (the number $\pi\gamma'$ is the angle of the projections of D_2 and D_3 on the horizontal plane, except if D_3 is vertical, in what case it can take any value).

The number $\pi\alpha_0$ is the geometric angle of $\vec{u}_1$ and $-\vec{u}_2$. Let us denote by $\pi\beta_0$ with $\beta_0 \in (0,1)$ the geometric angle of $\vec{u}_3$ and $-\vec{u}_1$, and by $\pi\gamma_0$ with $\gamma_0 \in (0,1)$ the geometric angle of $\vec{u}_2$ and $-\vec{u}_3$ (see figure 6).

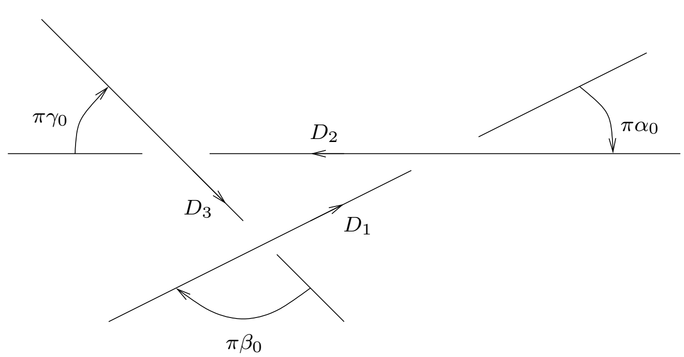

FIGURE 6. Three lines in generic position.

We denote by $\vec{v}_0$, $\vec{v}_1$ and $\vec{v}_\infty$ the vectors associated to (D_1, D_2), (D_2, D_3) and (D_3, D_1) (see definition 2):

$$\vec{v}_0 = -\frac{\vec{u}_1 \times \vec{u}_2}{||\vec{u}_1 \times \vec{u}_2||} = -\vec{e}_3, \quad \vec{v}_1 = -\frac{\vec{u}_2 \times \vec{u}_3}{||\vec{u}_2 \times \vec{u}_3||}, \quad \vec{v}_\infty = -\frac{\vec{u}_3 \times \vec{u}_1}{||\vec{u}_3 \times \vec{u}_1||}.$$

We set $A = -\mathrm{D}(D_1, D_2)$, $B = -\mathrm{D}(D_3, D_1)$ and $C = -\mathrm{D}(D_2, D_3)$. Finally we denote by ε_0 the sign of $\det(\vec{u}_1, \vec{u}_2, \vec{u}_3)$.

Proposition 9. *The map*

$$L : (D_1, D_2, D_3) \mapsto (\alpha_0, \gamma_0, \beta_0, -A, -C, -B, \varepsilon_0)$$

is a bijection from $\mathcal{D}$ *onto* $\mathcal{K} \times \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}^* \times \{1, -1\}$ *where* $\mathcal{K}$ *is the set of the triples* $(\alpha_0, \gamma_0, \beta_0) \in \mathbb{R}^3$ *satisfying*

$$\begin{aligned} \alpha_0 + \beta_0 + \gamma_0 > 1, \quad -\alpha_0 + \beta_0 + \gamma_0 < 1, \\ \alpha_0 - \beta_0 + \gamma_0 < 1, \quad \alpha_0 + \beta_0 - \gamma_0 < 1. \end{aligned} \tag{2}$$

Proof. The fact that $(\alpha_0, \gamma_0, \beta_0) \in \mathcal{K}$ is a consequence of Gauss-Bonnet formula applied to the spherical triangles on $\mathbb{S}^2$ bounded by the circles $D_1^\perp$, $D_2^\perp$ and $D_3^\perp$.

Conversely, let $(\alpha_0, \gamma_0, \beta_0) \in \mathcal{K}$, $A, B, C \in \mathbb{R}^*$ and $\varepsilon_0 \in \{1, -1\}$. Then there exists a spherical triangle of angles $\pi\alpha_0$, $\pi\beta_0$ and $\pi\gamma_0$. The three corresponding oriented circles define unit vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ uniquely up to a direct isometry of $\mathbb{R}^3$. If the sign of $\det(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is not equal to ε_0, then we replace these vectors by their images by an indirect isometry of $\mathbb{R}^3$ (which does not change the angles of the spherical triangle). Up to now, $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ are uniquely determined.

Now we consider three lines D_1, D_2 and D_3 in $\mathbb{R}^3$ oriented by $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$. We translate D_2 in the direction of $\vec{u}_1 \times \vec{u}_2$ in order that $\mathrm{D}(D_1, D_2) = -A$. Then we translate D_3 in the direction of $\vec{u}_1 \times \vec{u}_3$ in order that $\mathrm{D}(D_3, D_1) = -B$. Finally we translate D_3 in the direction of $\vec{u}_1$ in order that $\mathrm{D}(D_2, D_3) = -C$ (this operation does not change $\mathrm{D}(D_3, D_1)$). The lines D_1, D_2 and D_3 are determined uniquely up to a direct isometry of $\mathbb{R}^3$. This completes the proof. ☐

A precise study of the space of triples of lines in generic position, and in particular a more detailed proof of proposition 9, can be found in [Bal03].

Definition 10. *Two triples in* $\mathcal{D}$ *are called dual configurations if their parameters only differ by the sign of* ε_0.

The dual configuration of that of figure 6 is shown on figure 7; in both configurations the lines D_1 and D_2 are horizontal, but D_3 "goes down" on figure 6 and "goes up" on figure 7.

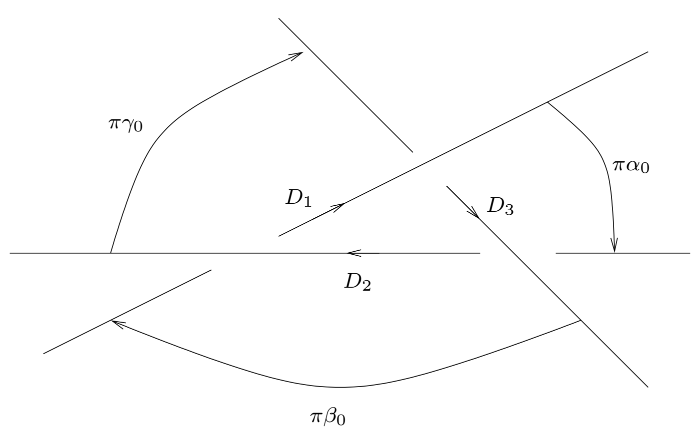

FIGURE 7. The dual configuration of that of figure 6.

Remark 11. *An indirect isometry of* $\mathbb{R}^3$ *changes* A, B, C *and* ε_0 *into their opposites. The dual configuration of a triple is not the image of this triple by a symmetry, but the directions of the straight lines are symmetric.*

We set $\beta' = 1 - \alpha - \gamma'$ (the number $\pi\beta'$ is the angle of the projections of D_1 and D_3 on the horizontal plane, except if D_3 is vertical, in what case it can take any value).

Since $\vec{v}_1$ and $-\vec{e}_3$ are normal to the x_1-axis, there exists a rotation R about the oriented x_1-axis that maps $\vec{v}_1$ onto $-\vec{e}_3$; we denote by θ its angle. We have $R(\vec{u}_2)=\vec{u}_2$ and $R(\vec{u}_3)=(\cos(\pi\gamma_0),-\sin(\pi\gamma_0),0)$. In the same way, $\vec{v}_\infty$ and $-\vec{e}_3$ are normal to D_1, so there exists a rotation $\hat{R}$ about the oriented line D_1 that maps $\vec{v}_\infty$ onto $-\vec{e}_3$; we denote by $\hat{\theta}$ its angle. We denote by T the rotation of angle $-\pi\alpha_0$ with respect to the x_3-axis. We have $T\circ\hat{R}(\vec{u}_1)=\vec{e}_1$, $T\circ\hat{R}(\vec{u}_3)=(\cos(\pi\beta_0),-\sin(\pi\beta_0),0)$ and $T\circ\hat{R}(\vec{v}_\infty)=-\vec{e}_3$.

Finally we set $t=\tan\frac{\theta}{2}$ and $\hat{t}=\tan\frac{\hat{\theta}}{2}$.

Lemma 12. *We have*

$$\cos\theta=\frac{\cos(\pi\beta_0)+\cos(\pi\alpha_0)\cos(\pi\gamma_0)}{\sin(\pi\alpha_0)\sin(\pi\gamma_0)},$$
$$\cos\hat{\theta}=\frac{\cos(\pi\gamma_0)+\cos(\pi\alpha_0)\cos(\pi\beta_0)}{\sin(\pi\alpha_0)\sin(\pi\beta_0)}.$$

Proof. We notice that the numbers $\sin(\pi\alpha_0)$, $\sin(\pi\beta_0)$ and $\sin(\pi\gamma_0)$ are positive.

We have

$$\cos(\pi\gamma_0)=<\vec{u}_2,-\vec{u}_3>=\cos(\pi\gamma')\sin\kappa,$$
$$\sin(\pi\gamma_0)=\sqrt{1-\cos^2(\pi\gamma_0)}=\sqrt{\sin^2(\pi\gamma')\sin^2\kappa+\cos^2\kappa},$$

$$\begin{aligned}\cos(\pi\beta_0)&=<\vec{u}_3,-\vec{u}_1>\\&=-\cos(\pi\alpha_0)\cos(\pi\gamma')\sin\kappa+\sin(\pi\alpha_0)\sin(\pi\gamma')\sin\kappa.\end{aligned}$$

We compute that

$$\vec{u}_1\times\vec{u}_2=\sin(\pi\alpha_0)\vec{e}_3,\quad \vec{u}_2\times\vec{u}_3=(0,\cos\kappa,\sin(\pi\gamma')\sin\kappa).$$

Thus we have

$$\cos\theta=\frac{<\vec{u}_1\times\vec{u}_2,\vec{u}_2\times\vec{u}_3>}{||\vec{u}_1\times\vec{u}_2||\cdot||\vec{u}_2\times\vec{u}_3||}=\frac{\sin(\pi\gamma')\sin\kappa}{\sqrt{\sin^2(\pi\gamma')\sin^2\kappa+\cos^2\kappa}}.$$

Finally we get

$$\cos(\pi\beta_0)=-\cos(\pi\alpha_0)\cos(\pi\gamma_0)+\sin(\pi\alpha_0)\sin(\pi\gamma_0)\cos\theta.$$

This proves the first formula.

And we have

$$\cos(\pi\beta_0)=\cos(\pi\beta')\sin\kappa,$$
$$\sin(\pi\beta_0)=\sqrt{1-\cos^2(\pi\beta_0)}=\sqrt{\sin^2(\pi\beta')\sin^2\kappa+\cos^2\kappa}.$$

We compute that

$$\vec{u}_3\times\vec{u}_1=(-\sin(\pi\alpha_0)\cos\kappa,\cos(\pi\alpha_0)\cos\kappa,\sin(\pi\beta')\sin\kappa).$$

Thus we have

$$\cos\hat{\theta}=\frac{<\vec{u}_1\times\vec{u}_2,\vec{u}_3\times\vec{u}_1>}{||\vec{u}_1\times\vec{u}_2||\cdot||\vec{u}_3\times\vec{u}_1||}=\frac{\sin(\pi\beta')\sin\kappa}{\sqrt{\sin^2(\pi\beta')\sin^2\kappa+\cos^2\kappa}}.$$

Finally we have

$$\begin{aligned}\cos(\pi\gamma_0)&=-\cos(\pi(\alpha_0+\beta'))\sin\kappa\\&=-\cos(\pi\alpha_0)\cos(\pi\beta')\sin\kappa+\sin(\pi\alpha_0)\sin(\pi\beta')\sin\kappa\\&=-\cos(\pi\alpha_0)\cos(\pi\beta_0)+\sin(\pi\alpha_0)\sin(\pi\beta_0)\cos\hat{\theta}.\end{aligned}$$

This proves the second formula. □

Lemma 13. *The signs of* $\cos\kappa$, $\sin\theta$, $\sin\hat{\theta}$, t *and* $\hat{t}$ *are equal to* ε_0.

Proof. Since $\sin\theta = \frac{2t}{1+t^2}$, t and $\sin\theta$ have the same sign. In the same way $\hat{t}$ and $\sin\hat{\theta}$ have the same sign.

By definition of θ we have $\vec{v}_1 \times (-\vec{e}_3) = \sin\theta\vec{e}_1$. We compute that

$$\vec{v}_1 \times (-\vec{e}_3) = \frac{\cos\kappa}{\sqrt{\sin^2(\pi\gamma')\sin^2\kappa + \cos^2\kappa}}\vec{e}_1,$$

so $\cos\kappa$ and $\sin\theta$ have the same sign.

In the same way we have $\vec{v}_\infty \times (-\vec{e}_3) = \sin\hat{\theta}\vec{u}_1$, so

$$\sin\hat{\theta} = \det(\vec{v}_\infty, -\vec{e}_3, \vec{u}_1) = \frac{\cos\kappa}{\sqrt{\sin^2(\pi\beta')\sin^2\kappa + \cos^2\kappa}}.$$

Finally we have $\det(\vec{u}_1, \vec{u}_2, \vec{u}_3) = \sin(\pi\alpha_0)\cos\kappa$. $\square$

3.2. **The Hopf differential and the spinor data.** Proceeding as for lemma 6, we get the following result.

Lemma 14. *Let $x : \Sigma \to \mathbb{R}^3$ be a conformal minimal immersion bounded by three straight lines. Let (g,ω) be its Weierstrass data. Then its Hopf differential Q extends to a holomorphic 2-form on $\mathbb{C}\setminus\{0,1\}$ and the Schwarzian derivative $\mathrm{S}_z g$ of its Gauss map extends to a meromorphic 2-form on $\mathbb{C}\setminus\{0,1\}$.*

From now on we consider a triple of lines $(D_1, D_2, D_3) \in \mathcal{D}$. Let

$$L(D_1, D_2, D_3) = (\alpha_0, \gamma_0, \beta_0, -A, -C, -B, \varepsilon_0),$$

$\alpha \in \alpha_0 + 2\mathbb{Z}$, $\beta \in \beta_0 + 2\mathbb{Z}$ and $\gamma \in \gamma_0 + 2\mathbb{Z}$. We assume that $x : \Sigma \to \mathbb{R}^3$ is a conformal minimal immersion bounded by (D_1, D_2, D_3) and having helicoidal ends of parameters (A,α), (B,β) and (C,γ) at 0, ∞ and 1 respectively. We denote by (g,ω) its Weierstrass data.

Proposition 15 (Riemann, [Rie68])**.** *Then the Hopf differential of x is*

$$Q = iz^{-2}(z-1)^{-2}\varphi(z)\mathrm{d}z^2 \tag{3}$$

where

$$\varphi(z) = \frac{B\beta}{2\pi}z(z-1) - \frac{A\alpha}{2\pi}(z-1) + \frac{C\gamma}{2\pi}z. \tag{4}$$

Proof. At $z=0$ we have $Q \sim i\frac{A\alpha}{2\pi}z^{-2}\mathrm{d}z^2$. At $z=1$ we have $Q \sim i\frac{C\gamma}{2\pi}(z-1)^{-2}\mathrm{d}z^2$. At $z=\infty$ we have $Q \sim i\frac{B\beta}{2\pi}z^{-2}\mathrm{d}z^2$. Hence the map $\varphi(z) = z^2(z-1)^2\frac{Q}{i\mathrm{d}z^2}$ has no singularity on $\mathbb{C}$, and we have $\varphi(z) = \mathrm{O}(z^2)$ when $z \to \infty$, so φ is a polynomial of degree less than or equal to 2. Finally we compute that φ has the announced expression. $\square$

Lemma 16. *The polynomial φ defined by (4) has two nonreal conjugate roots if and only if $A\alpha$, $B\beta$ and $C\gamma$ have the same sign, $\sqrt{|A\alpha|} < \sqrt{|B\beta|} + \sqrt{|C\gamma|}$, $\sqrt{|B\beta|} < \sqrt{|A\alpha|} + \sqrt{|C\gamma|}$ and $\sqrt{|C\gamma|} < \sqrt{|A\alpha|} + \sqrt{|B\beta|}$.*

It has a double real root if and only if $A\alpha$, $B\beta$ and $C\gamma$ have the same sign, and $\sqrt{|A\alpha|} = \sqrt{|B\beta|} + \sqrt{|C\gamma|}$, $\sqrt{|B\beta|} = \sqrt{|A\alpha|} + \sqrt{|C\gamma|}$ or $\sqrt{|C\gamma|} = \sqrt{|A\alpha|} + \sqrt{|B\beta|}$.

It has two distinct real roots in all other cases.

Proof. The discriminant of φ is $\frac{\delta}{4\pi^2}$ where

$$\delta = A^2\alpha^2 + B^2\beta^2 + C^2\gamma^2 - 2AB\alpha\beta - 2AC\alpha\gamma - 2BC\beta\gamma.$$

Thus the three cases in the lemma correspond respectively to $\delta < 0$, $\delta = 0$ and $\delta > 0$. The expression δ is a polynomial in the variable C, whose discriminant is equal to $16AB\alpha\beta\gamma^2$. If $A\alpha B\beta < 0$, then $\delta(C) > 0$ for all $C \in \mathbb{R}^*$.

We assume that $A\alpha > 0$ and $B\beta > 0$. Then we have $\delta(C) < 0$ if and only if
$$(\sqrt{A\alpha} - \sqrt{B\beta})^2 < C\gamma < (\sqrt{A\alpha} + \sqrt{B\beta})^2.$$
This condition is not satisfied if $C\gamma < 0$, and if $C\gamma > 0$ it is satisfied if and only if $\sqrt{C\gamma} > \sqrt{A\alpha} - \sqrt{B\beta}$, $\sqrt{C\gamma} > \sqrt{B\beta} - \sqrt{A\alpha}$ and $\sqrt{C\gamma} < \sqrt{A\alpha} + \sqrt{B\beta}$. And we have $\delta(C) = 0$ if and only if $C > 0$, and $\sqrt{C\gamma} = \sqrt{A\alpha} - \sqrt{B\beta}$, $\sqrt{C\gamma} = \sqrt{B\beta} - \sqrt{A\alpha}$ or $\sqrt{C\gamma} = \sqrt{A\alpha} + \sqrt{B\beta}$.

We deal with the case where $A\alpha < 0$ and $B\beta < 0$ in the same way. □

We set
$$\zeta(z) = \int_0^z \varphi(\tau)\mathrm{d}\tau.$$
This is a local diffeomorphism except at the zeros of φ. The Schwarzian derivative of g in the "coordinate" ζ satisfies
$$\mathrm{S}_\zeta g = \mathrm{S}_z g - \mathrm{S}_z \zeta.$$
We set
$$\begin{aligned}\Theta = \frac{\mathrm{S}_\zeta g}{\mathrm{d}z^2} &= \varphi\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)' - \frac{\varphi^2}{2}\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)^2 \\ &= \varphi^2\left(\left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 - \left(\frac{\varphi''}{\varphi}\right)' + \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2\right).\end{aligned}$$

Let (ξ_1, ξ_2) be the spinor data of x. We set
$$\xi_1 = z^{-1}(z-1)^{-1}k_1\sqrt{\mathrm{d}z}, \quad \xi_2 = z^{-1}(z-1)^{-1}k_2\sqrt{\mathrm{d}z}.$$
Then k_1 and k_2 are holomorphic functions on Σ, and we have
$$\begin{gathered} g = \tfrac{k_2}{k_1}, \quad \omega = z^{-2}(z-1)^{-2}k_1^2\mathrm{d}z, \\ Q = z^{-2}(z-1)^{-2}(k_1k_2' - k_1'k_2)\mathrm{d}z^2. \end{gathered} \tag{5}$$
We will abusively call k_1 and k_2 the spinors associated to (g, ω).

Lemma 17 (Riemann, [Rie68]). *The functions k_1 and k_2 satisfy the following relation on Σ:*
$$k_1k_2' - k_1'k_2 = i\varphi. \tag{6}$$
They are solutions on Σ of the following differential equation:
$$k'' - \frac{\varphi'}{\varphi}k' + \frac{\Theta}{2}k = 0. \tag{7}$$

Proof. Equality (6) follows from (3).

Since $k_1^2 = \frac{i\varphi}{g'}$, we have
$$2\frac{k_1'}{k_1} = \frac{\varphi'}{\varphi} - \frac{g''}{g'}.$$
On the other hand we have
$$\frac{1}{g'}\left(\frac{g'}{\varphi}\right)' = \frac{1}{g'}\left(\frac{g''}{\varphi} - \frac{\varphi'g'}{\varphi^2}\right) = -2\frac{k_1'}{\varphi k_1}.$$
So
$$\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)' = -2\frac{k_1''}{\varphi k_1} + 2\frac{{k_1'}^2}{\varphi k_1^2} + 2\frac{\varphi'k_1'}{\varphi^2 k_1},$$
and
$$\Theta = \varphi\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)' - \frac{\varphi^2}{2}\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)^2 = -2\frac{k_1''}{k_1} + 2\frac{\varphi'k_1'}{\varphi k_1}.$$

Thus k_1 is solution of equation (7).

We also have

$$2\frac{k_2'}{k_2} = 2\frac{g'}{g} + \frac{\varphi'}{\varphi} - \frac{g''}{g'}.$$

So

$$\frac{1}{g'}\left(\frac{g'}{\varphi}\right)' = 2\frac{g'}{\varphi g} - 2\frac{k_2'}{\varphi k_2},$$

$$\left(\frac{1}{g'}\left(\frac{g'}{\varphi}\right)'\right)' = 2\frac{g''}{\varphi g} - 2\frac{g'^2}{\varphi g^2} - 2\frac{\varphi' g'}{\varphi^2 g} - 2\frac{k_2''}{\varphi k_2} + 2\frac{{k_2'}^2}{\varphi k_2^2} + 2\frac{\varphi' k_2'}{\varphi^2 k_2},$$

$$\Theta = 2\frac{g''}{g} - 4\frac{g'^2}{g^2} - 2\frac{\varphi' g'}{\varphi g} - 2\frac{k_2''}{k_2} + 2\frac{\varphi' k_2'}{\varphi k_2} + 4\frac{g' k_2'}{g k_2} = -2\frac{k_2''}{k_2} + 2\frac{\varphi' k_2'}{\varphi k_2}.$$

Thus k_2 is solution of equation (7). □

Lemma 18. *We have*

$$\Theta(z) = \frac{\Phi(z)}{z^2(z-1)^2} + \frac{\Lambda(z)}{z(z-1)\varphi(z)} + \frac{2\varphi''}{\varphi(z)}$$

where

$$\Phi(z) = \frac{1-\beta^2}{2}z(z-1) - \frac{1-\alpha^2}{2}(z-1) + \frac{1-\gamma^2}{2}z \tag{8}$$

and where Λ is an affine function.

Proof. The form $\mathrm{S}_z\zeta$ is meromorphic on $\bar{\mathbb{C}}$, with double poles at ∞ and the roots of φ. Since $\Theta \mathrm{d}z^2 = \mathrm{S}_z g - \mathrm{S}_z\zeta$, by section 2.2 and lemma 14 the function Θ is meromorphic on $\bar{\mathbb{C}}$, its possible poles are 0, 1, ∞ and the zeros of φ, and these poles are at most double.

By lemma 7 we have $\mathrm{S}_z g \sim \frac{1-\alpha^2}{2}z^{-2}\mathrm{d}z^2$ when $z \to 0$. On the other hand, $\mathrm{S}_z\zeta$ is holomorphic at 0 since 0 is not a zero of φ. Thus we have $\Theta(z) \sim \frac{1-\alpha^2}{2}z^{-2}$ when $z \to 0$. In the same way we have $\Theta(z) \sim \frac{1-\gamma^2}{2}(z-1)^{-2}$ when $z \to 1$ and $\Theta(z) \sim \frac{9-\beta^2}{2}z^{-2}\mathrm{d}z^2$ when $z \to \infty$ (since $\mathrm{S}_z\zeta \sim -4z^{-2}\mathrm{d}z^2$). At a root of φ, since g and φ have branch points of the same order, the order -2 terms in $\mathrm{S}_z g$ and $\mathrm{S}_z\zeta$ at this root are equal, and so the order of Θ is greater than or equal to -1.

We have $\frac{2\varphi''}{\varphi(z)} \sim 4z^{-2}$ when $z \to \infty$ (and φ'' is a constant). Consequently the function $\Lambda = z(z-1)\varphi(\Theta - z^{-2}(z-1)^{-2}\Phi) - 2z(z-1)\varphi''$ is holomorphic on $\mathbb{C}$, and we have $\Lambda(z) = \mathrm{O}(z)$ when $z \to \infty$, so it is an affine function. □

Lemma 19. *If the roots of φ are distinct, then there are at most four possibilities for the function Λ.*

Proof. Equation (7) has regular singularities at 0, 1, ∞ and the roots a_1 and a_2 of φ (see [WW63], paragraph 10.3). Moreover, at least one of the roots of φ lies in Σ, for example a_1.

Since $a_1 \neq a_2$, the exponents of equation (7) at a_1 are 0 and 2, and since k_1 and k_2 are well-defined on Σ, the solutions of (7) have no logarithmic term at a_1. Thus, in the neighbourhood of a_1, equation (7) has a solution having the following form:

$$\sum_{n=0}^{\infty} \lambda_n (z-a_1)^n$$

with $\lambda_0 \neq 0$. Writing $\Theta(z) = \psi_{-1}(z-a_1)^{-1}+\psi_0+\mathrm{O}(z-a_1)$, since $\frac{\varphi'(z)}{\varphi(z)} = \frac{1}{z-a_1}+\frac{1}{a_1-a_2}+\mathrm{O}(z-a_1)$, reporting in equation (7) we get

$$-\lambda_1 + \frac{\psi_{-1}}{2}\lambda_0 = 0, \quad \left(\frac{\psi_{-1}}{2} - \frac{1}{a_1-a_2}\right)\lambda_1 + \frac{\psi_0}{2}\lambda_0 = 0.$$

Hence we get

$$\psi_{-1}\left(\frac{\psi_{-1}}{2} - \frac{1}{a_1-a_2}\right) + \psi_0 = 0. \tag{9}$$

We also compute using (9) that

$$\Lambda(z) = \frac{B\beta}{2\pi}(-4a_1(a_1-1) + m_1\psi_{-1}) + \frac{B\beta}{2\pi}(m_2 + m_3\psi_{-1} + m_4\psi_{-1}^2)(z-a_1)$$

with

$$m_1 = a_1(a_1-1)(a_1-a_2), \quad m_2 = -\frac{(a_1-a_2)\Phi(a_1)}{a_1(a_1-1)} - 4(2a_1-1),$$
$$m_3 = 2a_1(a_1-1) + (2a_1-1)(a_1-a_2), \quad m_4 = -\frac{m_1}{2}.$$

Assume that a_1 and a_2 are distinct real roots. Then $a_2 \in \Sigma$ and we can apply the same argument at a_2 with a coefficient $\tilde{\psi}_{-1}$ analogous to ψ_{-1}: we have

$$\Lambda(z) = \frac{B\beta}{2\pi}(-4a_2(a_2-1) + \tilde{m}_1\tilde{\psi}_{-1}) + \frac{B\beta}{2\pi}(\tilde{m}_2 + \tilde{m}_3\tilde{\psi}_{-1} + \tilde{m}_4\tilde{\psi}_{-1}^2)(z-a_2)$$

where $\tilde{m}_j$ has the same expression as m_j exchanging a_1 and a_2. Identifying these two expressions of Λ, we get $\tilde{m}_2+\tilde{m}_3\tilde{\psi}_{-1}+\tilde{m}_4\tilde{\psi}_{-1}^2 = m_2+m_3\psi_{-1}+m_4\psi_{-1}^2$ and $m_1\psi_{-1} - \tilde{m}_1\tilde{\psi}_{-1} - 4a_1(a_1-1) + 4a_2(a_2-1) = (a_1-a_2)(m_2+m_3\psi_{-1}+m_4\psi_{-1}^2)$. Setting $R = a_1(a_1-1)\psi_{-1}$ and $\tilde{R} = a_2(a_2-1)\tilde{\psi}_{-1}$, we get

$$\left\{\begin{array}{rcl} 0 & = & -4 - \frac{\Phi(a_1)}{a_1(a_1-1)} + \left(\frac{1}{a_1-a_2} + \frac{1}{a_1} + \frac{1}{a_1-1}\right)R \\ & & \quad -\frac{1}{a_1-a_2}\tilde{R} - \frac{1}{2a_1(a_1-1)}R^2 \\ 0 & = & -4 - \frac{\Phi(a_2)}{a_2(a_2-1)} + \left(\frac{1}{a_2-a_1} + \frac{1}{a_2} + \frac{1}{a_2-1}\right)\tilde{R} \\ & & \quad -\frac{1}{a_2-a_1}R - \frac{1}{2a_2(a_2-1)}\tilde{R}^2. \end{array}\right. \tag{10}$$

Assume that a_1 and a_2 are complex conjugate roots. Since $\Theta(z) \in \mathbb{R}$ when $z \in \mathbb{R}\setminus\{0,1\}$, Λ must have real coefficients, so we have $\mathrm{Im}(m_2 + m_3\psi_{-1} + m_4\psi_{-1}^2) = 0$ and $\mathrm{Im}(-4a_1(a_1-1) + m_1\psi_{-1} - a_1(m_2+m_3\psi_{-1}+m_4\psi_{-1}^2)) = 0$. Setting $R = a_1(a_1-1)\psi_{-1}$ and $\tilde{R} = \bar{R}$, since $a_2 = \overline{a_1}$, this is equivalent to system (10).

System (10) has at most four solutions $(R, \tilde{R})$. Thus there are at most four possible functions Λ. □

Lemma 20. *If the polynomial φ has a double real root, then there are at most three possibilities for the function Λ.*

Proof. Equation (7) has regular singularities at 0, 1, ∞ and the root a_1 of φ. Moreover, a_1 lies in Σ. Since a_1 is a double root of φ, the exponents of equation (7) at a_1 are 0 and 3, and since k_1 and k_2 are well-defined on Σ, the solutions of (7) have no logarithmic term at a_1. Thus, in the neighbourhood of a_1, equation (7) has a solution having the following form:

$$\sum_{n=0}^{\infty}\lambda_n(z-a_1)^n$$

with $\lambda_0 \neq 0$. Writing

$$\Theta(z) = \psi_{-1}(z-a_1)^{-1} + \psi_0 + \psi_1(z-a_1) + \mathrm{O}((z-a_1)^2),$$

since $\frac{\varphi'(z)}{\varphi(z)} = \frac{2}{z-a_1}$, reporting in equation (7) we get

$$-2\lambda_1 + \frac{\psi_{-1}}{2}\lambda_0 = 0, \quad -2\lambda_2 + \frac{\psi_{-1}}{2}\lambda_1 + \frac{\psi_0}{2}\lambda_0 = 0,$$
$$\frac{\psi_{-1}}{2}\lambda_2 + \frac{\psi_0}{2}\lambda_1 + \frac{\psi_1}{2}\lambda_0 = 0.$$

Hence we get

$$\frac{\psi_{-1}^3}{16} + \frac{\psi_{-1}\psi_0}{2} + \psi_1 = 0. \tag{11}$$

We also compute that

$$\begin{aligned}\Lambda(z) &= -4\frac{B\beta}{2\pi}a_1(a_1-1) + \frac{B\beta}{2\pi}n_1(z-a_1)\\ &\quad + \frac{B\beta}{2\pi}n_2(z-a_1)^2 + \frac{B\beta}{2\pi}n_3(z-a_1)^3 + \mathrm{O}((z-a_1)^4)\end{aligned}$$

with

$$n_1 = a_1(a_1-1)\psi_{-1} - 4(2a_1-1),$$
$$n_2 = a_1(a_1-1)\left(\psi_0 - \frac{\Phi(a_1)}{a_1^2(a_1-1)^2}\right) + (2a_1-1)\psi_{-1} - 4,$$

$$\begin{aligned}n_3 &= a_1(a_1-1)\psi_1 - \frac{\Phi'(a_1)}{a_1(a_1-1)} + 2\frac{\Phi(a_1)}{a_1(a_1-1)}\left(\frac{1}{a_1} + \frac{1}{a_1-1}\right)\\ &\quad + (2a_1-1)\left(\psi_0 - \frac{\Phi(a_1)}{a_1^2(a_1-1)^2}\right) + \psi_{-1}.\end{aligned}$$

On the other hand, Λ is an affine function, so we have $n_2 = n_3 = 0$, and using (11) we obtain that ψ_{-1} is solution of a degree 3 polynomial equation. Thus there are at most three possible functions Λ. □

All that has been done up to now does not depend on ε_0, *i.e.* it holds for (D_1, D_2, D_3) as well as for its dual configuration.

Theorem 21. *There exist at most four minimal disks bounded by (D_1, D_2, D_3) or its dual configuration and with helicoidal ends of parameters (A, α), (B, β) and (C, γ) at 0, ∞ and 1 respectively.*

Proof. By lemmas 19 and 20, it suffices to prove that for each possibility of the Schwarzian derivative there exists at most one minimal immersion bounded by (D_1, D_2, D_3) or its dual configuration and with helicoidal ends of parameters (A, α), (B, β), (C, γ).

Assume that the function Θ is known. Then the set of the solutions of equation (7) on Σ is a vector space generated by two independent solutions. Thus, if (g, ω) and $(\tilde{g}, \tilde{\omega})$ are the Weierstrass data of two minimal immersions corresponding to Θ, then g and $\tilde{g}$ are quotients of linear combinations of these two independent solutions, so there exists a Möbius transform $\mu : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ such that $\tilde{g} = \mu \circ g$. But the value of the Gauss map at each end is uniquely determined, so we have $g(0) = \tilde{g}(0)$, $g(1) = \tilde{g}(1)$ and $g(\infty) = \tilde{g}(\infty)$. Moreover, $g(0)$, $g(1)$ and $g(\infty)$ are pairwise distinct (since the straight lines do not lie in parallel planes), so μ is the identity, and so $\tilde{g} = g$ and $\tilde{\omega} = \omega$. □

3.3. **Some facts about the hypergeometric differential equation.** In this section we recall some facts about the hypergeometric differential equation and hypergeometric series that will be useful to give explicit examples of minimal disks bounded by three straight lines.

Let s_1, s_2 and s_3 be three complex numbers such that $s_3 \notin -\mathbb{N}^*$. The hypergeometric series is defined by

$$\mathrm{F}(s_1, s_2; s_3; z) = \frac{\Gamma(s_3)}{\Gamma(s_1)\Gamma(s_2)} \sum_{n=0}^{\infty} \frac{\Gamma(s_1+n)\Gamma(s_2+n)}{\Gamma(s_3+n)} z^n$$

for $|z| < 1$.

We use the notations of section 3.2. We define eight numbers

$$s_{\pm\pm\pm} = \frac{1 \pm \alpha \pm \beta \pm \gamma}{2}. \tag{12}$$

These numbers are noninteger. We also set

$$\begin{aligned} \Pi \; = \; & (1+\alpha+\beta+\gamma)(1-\alpha+\beta+\gamma)(1+\alpha-\beta+\gamma) \\ & \times(1+\alpha+\beta-\gamma)(1-\alpha-\beta+\gamma)(1-\alpha+\beta-\gamma) \\ & \times(1+\alpha-\beta-\gamma)(1-\alpha-\beta-\gamma). \end{aligned} \tag{13}$$

We consider the following hypergeometric equation on Σ:

$$w'' + \left(\frac{1-\alpha}{z} + \frac{1-\gamma}{z-1}\right) w' + s_{---}s_{-+-}\frac{w}{z(z-1)} = 0. \tag{14}$$

This equation is usually denoted by $\mathrm{P}\left\{\begin{array}{cccc} 0 & s_{---} & 0 & \\ \alpha & s_{-+-} & \gamma & z \end{array}\right\}$. Since we have $\alpha, \beta, \gamma \notin \mathbb{Z}$, the fundamental system of linear independent solutions of hypergeometric equation (14) at the singular points is given (see section 2.2, paragraph 1 in [MOS66], or [BPS02])

- on Σ_0 by

$$w_1^{(0)}(z) = \mathrm{F}(s_{---}, s_{-+-}; 1-\alpha; z),$$
$$w_2^{(0)}(z) = z^\alpha\, \mathrm{F}(s_{+--}, s_{++-}; 1+\alpha; z),$$

- on Σ_1 by

$$\begin{aligned} w_1^{(1)}(z) &= \mathrm{F}(s_{---}, s_{-+-}; 1-\gamma; 1-z) \\ &= z^\alpha\, \mathrm{F}(s_{+--}, s_{++-}; 1-\gamma; 1-z), \end{aligned}$$

$$\begin{aligned} w_2^{(1)}(z) &= (1-z)^\gamma\, \mathrm{F}(s_{--+}, s_{-++}; 1+\gamma; 1-z) \\ &= z^\alpha(1-z)^\gamma\, \mathrm{F}(s_{+-+}, s_{+++}; 1+\gamma; 1-z), \end{aligned}$$

- on Σ_∞ by

$$w_1^{(\infty)}(z) = z^{-s_{---}}\, \mathrm{F}(s_{---}, s_{+--}; 1-\beta; z^{-1}),$$
$$w_2^{(\infty)}(z) = z^{-s_{-+-}}\, \mathrm{F}(s_{-+-}, s_{++-}; 1+\beta; z^{-1}).$$

The second expressions for $w_1^{(1)}$ and $w_2^{(1)}$ are obtained using first formula of section 2.4.1 in [MOS66].

For $z \in (-1, 0)$ we have $w_1^{(0)}(z) \in \mathbb{R}$ and $w_2^{(0)}(z) \in e^{i\pi\alpha}\mathbb{R}$; for $z \in (0,1)$ we have $w_1^{(0)}(z) \in \mathbb{R}$ and $w_2^{(0)}(z) \in \mathbb{R}$. For $z \in (0,1)$ we have $w_1^{(1)}(z) \in \mathbb{R}$ and $w_2^{(1)}(z) \in \mathbb{R}$; for $z \in (1,2)$ we have $w_1^{(1)}(z) \in \mathbb{R}$ and $w_2^{(1)}(z) \in e^{-i\pi\gamma}\mathbb{R}$. For $z \in (1,+\infty)$ we have $w_1^{(\infty)}(z) \in \mathbb{R}$ and $w_2^{(\infty)}(z) \in \mathbb{R}$; for $z \in (-\infty, -1)$ we have $w_1^{(\infty)}(z) \in e^{-i\pi s_{---}}\mathbb{R}$ and $w_2^{(\infty)}(z) \in e^{-i\pi s_{-+-}}\mathbb{R}$.

These solutions are connected in the following way. On $\Sigma_0 \cap \Sigma_1$ we have

$$\begin{pmatrix} w_1^{(0)} \\ w_2^{(0)} \end{pmatrix} = \nu \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \end{pmatrix}$$

where

$$\nu = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(1-\alpha)\Gamma(\gamma)}{\Gamma(s_{--+})\Gamma(s_{-++})} & \frac{\Gamma(1-\alpha)\Gamma(-\gamma)}{\Gamma(s_{---})\Gamma(s_{-+-})} \\ \frac{\Gamma(1+\alpha)\Gamma(\gamma)}{\Gamma(s_{+-+})\Gamma(s_{+++})} & \frac{\Gamma(1+\alpha)\Gamma(-\gamma)}{\Gamma(s_{+--})\Gamma(s_{++-})} \end{pmatrix} \tag{15}$$

(we used fourth formula of section 2.4.1 in [MOS66] to compute this matrix). In the same way we have

$$\begin{pmatrix} w_1^{(0)} \\ w_2^{(0)} \end{pmatrix} = \hat{\nu} \begin{pmatrix} w_1^{(\infty)} \\ w_2^{(\infty)} \end{pmatrix}$$

where

$$\begin{aligned} \hat{\nu} &= \begin{pmatrix} \hat{\nu}_{11} & \hat{\nu}_{12} \\ \hat{\nu}_{21} & \hat{\nu}_{22} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\pi s_{---}} \frac{\Gamma(1-\alpha)\Gamma(\beta)}{\Gamma(s_{-+-})\Gamma(s_{-++})} & e^{i\pi s_{-+-}} \frac{\Gamma(1-\alpha)\Gamma(-\beta)}{\Gamma(s_{---})\Gamma(s_{--+})} \\ e^{i\pi s_{+--}} \frac{\Gamma(1+\alpha)\Gamma(\beta)}{\Gamma(s_{++-})\Gamma(s_{+++})} & e^{i\pi s_{++-}} \frac{\Gamma(1+\alpha)\Gamma(-\beta)}{\Gamma(s_{+--})\Gamma(s_{+-+})} \end{pmatrix} \end{aligned} \tag{16}$$

(we used fifth formula of section 2.4.1 in [MOS66] to compute this matrix; we should notice that in this formula $(-z)^{-a}$ is defined with $\arg(-z) \in (-\pi, \pi)$, and thus with this convention we get $(-z)^{-a} = e^{i\pi a} z^{-a}$). This last formula is actually valid for the analytic continuations of the solutions on Σ_0 and Σ_∞, since the intersection of these two domains is empty.

In the sequel, σ_1 and σ_2 will denote the solutions on Σ of hypergeometric equation (14) such that

$$\sigma_1 = w_1^{(0)}, \quad \sigma_2 = w_2^{(0)}$$

on Σ_0.

Lemma 22. *We have $\sigma_1\sigma_2' - \sigma_1'\sigma_2 = \alpha z^{\alpha-1}(1-z)^{\gamma-1}$.*

Proof. Since σ_1 and σ_2 are solutions of (14), we get that $\sigma_1\sigma_2' - \sigma_1'\sigma_2$ is a solution on Σ of the following equation:

$$w' = -\left(\frac{1-\alpha}{z} + \frac{1-\gamma}{z-1}\right) w.$$

Thus it is proportional to $z^{\alpha-1}(1-z)^{\gamma-1}$. And since $\sigma_1\sigma_2' - \sigma_1'\sigma_2 \sim \alpha z^{\alpha-1}$ when $z \to 0$, we get the announced expression. □

Lemma 23. *We have $\nu_{12}\nu_{21} = -t^2\nu_{11}\nu_{22}$ where t has been defined in section 3.1.*

Proof. Using that $\Gamma(1+z) = z\Gamma(z)$ and $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ (see section 1.1 in [MOS66]), we compute that

$$\begin{aligned} \nu_{12}\nu_{21} &= \frac{\Gamma(1-\alpha)\Gamma(-\gamma)\Gamma(1+\alpha)\Gamma(\gamma)}{\Gamma(s_{---})\Gamma(s_{-+-})\Gamma(s_{+-+})\Gamma(s_{+++})} \\ &= -\frac{\alpha}{\gamma}\frac{\Gamma(1-\alpha)\Gamma(1-\gamma)\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(s_{---})\Gamma(s_{-+-})\Gamma(1-s_{-+-})\Gamma(1-s_{---})} \\ &= -\frac{\alpha}{\gamma}\frac{\sin(\pi s_{---})\sin(\pi s_{-+-})}{\sin(\pi\alpha)\sin(\pi\gamma)}, \end{aligned}$$

$$\begin{aligned}\nu_{11}\nu_{22} &= \frac{\Gamma(1-\alpha)\Gamma(\gamma)\Gamma(1+\alpha)\Gamma(-\gamma)}{\Gamma(s_{--+})\Gamma(s_{-++})\Gamma(s_{+--})\Gamma(s_{++-})} \\ &= -\frac{\alpha}{\gamma}\frac{\Gamma(1-\alpha)\Gamma(\gamma)\Gamma(\alpha)\Gamma(1-\gamma)}{\Gamma(s_{--+})\Gamma(s_{-++})\Gamma(1-s_{-++})\Gamma(1-s_{--+})} \\ &= -\frac{\alpha}{\gamma}\frac{\sin(\pi s_{--+})\sin(\pi s_{-++})}{\sin(\pi\alpha)\sin(\pi\gamma)}.\end{aligned}$$

Thus proving that $\nu_{12}\nu_{21} = -t^2\nu_{11}\nu_{22}$ is equivalent to prove that

$$\sin(\pi s_{---})\sin(\pi s_{-+-}) = -t^2\sin(\pi s_{--+})\sin(\pi s_{-++}). \tag{17}$$

But we have

$$\begin{aligned}2\sin(\pi s_{---})\sin(\pi s_{-+-}) &= \cos(\pi\beta) - \cos(\pi(-\alpha-\gamma+1)) \\ &= \cos(\pi\beta) + \cos(\pi\alpha)\cos(\pi\gamma) - \sin(\pi\alpha)\sin(\pi\gamma),\end{aligned}$$

$$\begin{aligned}2\sin(\pi s_{--+})\sin(\pi s_{-++}) &= \cos(\pi\beta) - \cos(\pi(-\alpha+\gamma+1)) \\ &= \cos(\pi\beta) + \cos(\pi\alpha)\cos(\pi\gamma) + \sin(\pi\alpha)\sin(\pi\gamma).\end{aligned}$$

Thus, since $\frac{1-t^2}{1+t^2} = \cos\theta$, condition (17) is equivalent to

$$\cos\theta = \frac{\cos(\pi\beta) + \cos(\pi\alpha)\cos(\pi\gamma)}{\sin(\pi\alpha)\sin(\pi\gamma)},$$

which is satified according to lemma 12 and since $\pi\alpha$, $\pi\beta$ and $\pi\gamma$ are congruent to $\pi\alpha_0$, $\pi\beta_0$ and $\pi\gamma_0$ modulo 2π. □

Lemma 24. *We have $\hat\nu_{12}\hat\nu_{21} = -\hat t^2\hat\nu_{11}\hat\nu_{22}$ where $\hat t$ has been defined in section 3.1.*

Proof. Since $e^{i\pi s_{---}}e^{i\pi s_{++-}} = e^{i\pi(1-\gamma)} = e^{i\pi s_{-+-}}e^{i\pi s_{+--}}$, proceeding as in lemma 23 and exchanging the roles of β and γ, we obtain that the equality of the lemma is equivalent to

$$\cos\hat\theta = \frac{\cos(\pi\gamma) + \cos(\pi\alpha)\cos(\pi\beta)}{\sin(\pi\alpha)\sin(\pi\beta)}.$$

This condition is satified according to lemma 12 and since $\pi\alpha$, $\pi\beta$ and $\pi\gamma$ are congruent to $\pi\alpha_0$, $\pi\beta_0$ and $\pi\gamma_0$ modulo 2π. □

Lemma 25. *We have*

$$\frac{\hat t\hat\nu_{11}}{\hat\nu_{21}} = e^{-i\pi\alpha}\frac{t\nu_{11}}{\nu_{21}}.$$

Proof. Proving this equality is equivalent to prove that

$$t\sin(\pi s_{--+}) = \hat t\sin(\pi s_{-+-}).$$

We have computed that

$$t^2 = -\frac{\sin(\pi s_{---})\sin(\pi s_{-+-})}{\sin(\pi s_{--+})\sin(\pi s_{-++})}, \quad \hat t^2 = -\frac{\sin(\pi s_{---})\sin(\pi s_{--+})}{\sin(\pi s_{-+-})\sin(\pi s_{-++})}.$$

We deduce that $t^2\sin^2(\pi s_{--+}) = \hat t^2\sin^2(\pi s_{-+-})$. Since $\sin\theta = \frac{2t}{1+t^2}$ and $\sin\hat\theta = \frac{2\hat t}{1+\hat t^2}$, we deduce from lemma 13 that t and $\hat t$ have the same sign. So it now suffices to prove that $\sin(\pi s_{--+})$ and $\sin(\pi s_{-+-})$ have the same sign.

We have $2\sin(\pi s_{--+})\sin(\pi s_{-+-}) = \cos(\pi(\gamma-\beta)) - \cos(\pi(1-\alpha)) = \cos(\pi(\gamma_0-\beta_0)) - \cos(\pi(1-\alpha_0))$. We also have $1-\alpha_0 \in (0,1)$ and $\gamma_0-\beta_0 \in (-1,1)$, and by (2) we have $1-\alpha_0 > |\gamma_0-\beta_0|$, so we get $\cos(\pi(1-\alpha_0)) < \cos(\pi(\gamma_0-\beta_0))$ and $2\sin(\pi s_{--+})\sin(\pi s_{-+-}) > 0$. This proves the lemma. □

3.4. **Existence of a minimal surface bounded by three lines.** In this section we give explicit examples of minimal disks bounded by three lines in generic position. We use the notations of section 3.2.

Let a, b and c be three real numbers. For $j = 1, 2$ and $z \in \Sigma$, we set

$$K_j = z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}((a+bz)\sigma_j + cz(1-z)\sigma_j'). \tag{18}$$

These functions were introduced by Riemann in his memoir [Rie68] (where they are denoted by k_1 and k_2).

Lemma 26 (Riemann, [Rie68])**.** *The functions K_1 and K_2 satisfy*

$$K_1K_2' - K_1'K_2 = z^{1-\alpha}(1-z)^{1-\gamma}(\sigma_1\sigma_2' - \sigma_1'\sigma_2)F(z)$$

with

$$\begin{aligned} F(z) &= a(a+c\alpha)(1-z) + (a+b)(a+b-c\gamma)z \\ &\quad -(b+s_{---}c)(b+s_{-+-}c)z(1-z). \end{aligned} \tag{19}$$

Proof. Using the fact that σ_j is a solution of (14), we compute that

$$\begin{aligned} K_j' &= z^{-\frac{1+\alpha}{2}}(1-z)^{-\frac{1+\gamma}{2}} \times \\ &\quad ((n_1z^2 + n_2z(1-z) + n_3(1-z)^2)\sigma_j) \\ &\quad +(n_4z^2(1-z) + n_5z(1-z)^2)\sigma_j') \end{aligned}$$

with

$$n_1 = -\frac{1-\gamma}{2}(a+b), \quad n_2 = \frac{\gamma-\alpha}{2}a + \frac{3-\alpha}{2}b + cs_{---}s_{-+-},$$

$$n_3 = \frac{1-\alpha}{2}a, \quad n_4 = a+b-\frac{1+\gamma}{2}c, \quad n_5 = a + \frac{1+\alpha}{2}c.$$

Thus we get

$$\begin{aligned} K_1K_2' - K_1'K_2 &= z^{1-\alpha}(1-z)^{1-\gamma}(\sigma_1\sigma_2' - \sigma_1'\sigma_2) \times \\ &\quad (((a+b)n_4 - cn_1)z^2 \\ &\quad +(an_4 + (a+b)n_5 - cn_2)z(1-z) \\ &\quad +(an_5 - cn_3)(1-z)^2). \end{aligned}$$

The last factor in this expression is a polynomial of degree 2. We compute that its values at 0 and 1 and its degree 2 coefficient are the same as those of the polynomial F in the lemma. □

Corollary 27. *The functions K_1 and K_2 satisfy*

$$K_1K_2' - K_1'K_2 = \alpha F, \quad K_1K_2'' - K_1''K_2 = \alpha F', \tag{20}$$

with F as in (19).

Proof. The first formula comes from lemmas 22 and 26. We obtain the second one by differentiation. □

Lemma 28. *Let $\lambda_1, \lambda_2, \mu_1, \mu_2$ be complex numbers such that $\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1 \neq 0$. Then the spinors $k_1 = \lambda_1^{-1}K_1 + \lambda_2^{-1}K_2$ and $k_2 = \mu_1K_1 + \mu_2K_2$ define a conformal minimal immersion $x : \Sigma \to \mathbb{R}^3$, with possibly a singular point at the root of F when F has a double root. The immersion x has, up to a translation in $\mathbb{R}^3$, a helicoidal end bounded by D_1 and D_2 and of parameters (A, α) if and only if $\lambda_2 = \mu_1 = 0$, $\lambda_1 \in i\mathbb{R}^*$, $\mu_2 \in \mathbb{R}^*$ and $\alpha\lambda_1^{-1}\mu_2 a(a+\alpha c) = i\frac{A\alpha}{2\pi}$.*

Proof. The singularities of x correspond to the common zeros of k_1 and k_2, and thus to the common zeros of K_1 and K_2. Using (20) we conclude that a singular point of x is necessarily a double root of F.

Let (g,ω) be the Weierstrass data of x. We will use the expressions of σ_1 and σ_2 valid in Σ_0, that is $\sigma_1 = w_1^{(0)}$ and $\sigma_2 = w_2^{(0)}$. We have

$$g = \frac{\mu_1 K_1 + \mu_2 K_2}{\lambda_1^{-1} K_1 + \lambda_2^{-1} K_2},$$

$$k_1 k_2' - k_1' k_2 = (\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1)(K_1 K_2' - K_1' K_2) = \alpha(\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1) F$$

with F as in (19).

Assume that x has, up to a translation in $\mathbb{R}^3$, a helicoidal end bounded by D_1 and D_2 and of parameters (A,α). Then when $z \to 0$ we have $Q \sim i\frac{A\alpha}{2\pi} z^{-2}\mathrm{d}z^2$, so we get $i\frac{A\alpha}{2\pi} = \alpha(\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1)F(0) = \alpha(\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1)a(a+\alpha c)$. This implies in particular that $\lambda_1^{-1}\mu_2 - \lambda_2^{-1}\mu_1 \in i\mathbb{R}$, $a \neq 0$ and $a + \alpha c \neq 0$.

Then we have $K_1(z) \sim a z^{\frac{1-\alpha}{2}}$ and $K_2(z) \sim (a+\alpha c) z^{\frac{1+\alpha}{2}}$ when $z \to 0$. We must have $g(z) \sim i\rho z^\alpha$ for some $\rho \in \mathbb{R}^*$ (see the proof of lemma 7), since $\alpha \in (0,1) + 2\mathbb{Z}$. This implies that $\mu_1 = 0$ if $\alpha > 0$ and $\lambda_2 = 0$ if $\alpha < 0$.

We deal with the case where $\alpha > 0$. In this case we have $g = \frac{\mu_2 K_2}{\lambda_1^{-1}K_1 + \lambda_2^{-1}K_2} \sim \lambda_1\mu_2 \frac{a+\alpha c}{a} z^\alpha$, so $\lambda_1\mu_2 \in i\mathbb{R}$. And since we also have $\lambda_1^{-1}\mu_2 \in i\mathbb{R}$ (because $\lambda_2^{-1}\mu_1 = 0$), we get $\mu_2^2 \in \mathbb{R}$ and $\lambda_1^2 \in \mathbb{R}$. Moreover, we have $g(z) \in i\bar{\mathbb{R}}$ if $z \in (0,1)$ (since D_2 is the x_1-axis), and the functions K_1 and K_2 take real values on $(0,1)$, so we also have $\lambda_2\mu_2 \in i\mathbb{R}$, and so $\lambda_2^2 \in \mathbb{R}$. Since x maps $(-1,0)$ onto a straight line, and since $\mathrm{d}(x_1 + ix_2) = \bar{\omega} - g^2\omega = \overline{z^{-2}(z-1)^{-2}k_1^2\mathrm{d}z} - z^{-2}(z-1)^{-2}k_2^2\mathrm{d}z$, the argument of $\overline{k_1^2} - k_2^2$ must be constant on $(-1,0)$. We have $\overline{k_1^2} - k_2^2 = \lambda_1^{-2}\overline{K_1^2} + 2\overline{\lambda_1^{-1}\lambda_2^{-1}K_1K_2} + \lambda_2^{-2}\overline{K_2^2} - \mu_2^2 K_2^2$; on the other hand we have $\arg K_1(z) \equiv \pi\frac{1-\alpha}{2} \mod \pi$ and $\arg K_2(z) \equiv \pi\frac{1+\alpha}{2} \mod \pi$ if $z \in (-1,0)$, so we conclude that $\lambda_2 = 0$. Thus we have $\omega \sim \lambda_1^{-2}a^2 z^{-1-\alpha}\mathrm{d}z$ and $g^2\omega \sim \mu_2^2(a+\alpha c)^2 z^{-1+\alpha}\mathrm{d}z$. Since $x_1 \to +\infty$ when $z \to 0$ with z real and positive (by condition 2 in definition 4), we get $\lambda_1^2 < 0$ and $\mu_2^2 > 0$, which implies $\lambda_1 \in i\mathbb{R}^*$ and $\mu_2 \in \mathbb{R}^*$.

We proceed in the same way in the case where $\alpha < 0$.

Conversely, assume that $\lambda_2 = \mu_1 = 0$, $\lambda_1 \in i\mathbb{R}^*$, $\mu_2 \in \mathbb{R}^*$ and $\alpha\lambda_1^{-1}\mu_2 a(a+\alpha c) = i\frac{A\alpha}{2\pi}$. Then we have $\arg\overline{k_1^2} = \arg(-k_2^2) = \pi\alpha$ on $(-1,0)$ and $\arg\overline{k_1^2} = \arg(-k_2^2) = \pi$ on $(0,1)$, and we have $\mathrm{d}(x_1 + ix_2) = \bar{\omega} - g^2\omega \sim \lambda_1^{-2}a^2\overline{z^{-1-\alpha}\mathrm{d}z} - \mu_2^2(a+\alpha c)^2 z^{-1+\alpha}\mathrm{d}z$. This proves that x has an end bounded by two lines D_1' and D_2' that are parallel to D_1 and D_2 respectively. Moreover, we have $g(z) \sim \lambda_1\mu_2\frac{a+\alpha c}{a}z^\alpha$ and $Q \sim \alpha\lambda_1^{-1}\mu_2 a(a+\alpha c)z^{-2}\mathrm{d}z^2 = i\frac{A\alpha}{2\pi}z^{-2}\mathrm{d}z^2$, so x has a helicoidal end of parameters (A,α) by lemma 4. This finally implies that $\mathrm{D}(D_1', D_2') = -A = \mathrm{D}(D_1, D_2)$, and so D_1' and D_2' are the images of D_1 and D_2 by a translation in $\mathbb{R}^3$. □

In the sequel we will study the minimal immersion $x : \Sigma \to \mathbb{R}^3$ (with possibly a singularity at a double root of F) given by the spinors $k_1 = \lambda^{-1}K_1$ and $k_2 = \mu K_2$ for some $\lambda \in \mathbb{C}^*$ and some $\mu \in \mathbb{C}^*$. We first notice the following fact.

Remark 29. *For $\rho \in \mathbb{R}^*$, the transformation*

$$(a,b,c,\lambda,\mu) \to (\rho a, \rho b, \rho c, \rho\lambda, \rho^{-1}\mu)$$

does not change the Weierstrass data (g,ω) (and consequently does not change the immersion x), and changes $\lambda^{-1}\mu$ into $\rho^{-2}\lambda^{-1}\mu$. Thus, without loss of generality, we can assume that $|\lambda\mu^{-1}| = |\alpha|$.

Remark 30. *Replacing (λ,μ) by $(-i\lambda, i\mu)$ would change g into $-g$ and ω into $-\omega$. Thus the immersion x would be replaced by its image by the reflexion about the x_3-axis.*

Proposition 31. *Let λ and μ be two nonzero complex numbers such that $|\lambda\mu^{-1}| = |\alpha|$. Let $x : \Sigma \to \mathbb{R}^3$ be the conformal minimal immersion (with possibly a singularity at a double root of F) whose Weierstrass data are given by*

$$g = \frac{k_2}{k_1}, \quad \omega = z^{-2}(z-1)^{-2}k_1^2,$$

where

$$k_1 = \lambda^{-1}K_1, \quad k_2 = \mu K_2$$

with K_1 and K_2 as in formula (18).

Then, up to a translation, the immersion x maps $(-\infty, 0)$, $(0,1)$ and $(1,+\infty)$ to D_1, D_2 and D_3 respectively, and has helicoidal ends of parameters (A,α), (B,β) and (C,γ) at 0, ∞ and 1 respectively if and only if the following conditions hold:

1. *$\lambda = -\varepsilon i\alpha\mu$ where ε is the sign of $\varepsilon_0\frac{\nu_{11}}{\alpha\nu_{21}}$ with ν as defined in section 3.3,*
2. *$\varepsilon\alpha\mu^2 = \frac{t\nu_{11}}{\nu_{21}}$ (this implies in particular that μ is real),*
3. *$F = \varepsilon\varphi$ with φ as in (4) and F as in (19), i.e. the real numbers a, b and c satisfy*

$$\left\{\begin{array}{lcc} \varepsilon\frac{A\alpha}{2\pi} & = & a(a+\alpha c) \\ \varepsilon\frac{B\beta}{2\pi} & = & (b+s_{---}c)(b+s_{-+-}c) \\ \varepsilon\frac{C\gamma}{2\pi} & = & (a+b)(a+b-\gamma c). \end{array}\right. \tag{21}$$

Proof. We denote by Δ_1, Δ_2, Δ_3, Δ_1^∞, Δ_1^0, Δ_3^0 and Δ_3^∞ the images by x of $(-\infty,0)$, $(0,1)$, $(1,\infty)$, $(-\infty,-1)$, $(-1,0)$, $(1,2)$ and $(2,+\infty)$.

If x has, up to a translation in $\mathbb{R}^3$, a helicoidal end at 0 bounded by D_1 and D_2 and of parameters (A,α), then by lemma 28 we have $\lambda \in i\mathbb{R}^*$, $\mu \in \mathbb{R}^*$ and $\alpha\lambda^{-1}\mu a(a+\alpha c) = i\frac{A\alpha}{2\pi}$. Since $|\lambda\mu^{-1}| = |\alpha|$, we have $\lambda = -\varepsilon i\alpha\mu$ with $\varepsilon = \pm 1$, and so the first equality in (21) holds.

From now on we assume that

$$\lambda = -\varepsilon i\alpha\mu \tag{22}$$

with $\varepsilon = \pm 1$ and $\mu \in \mathbb{R}^*$, and that $\frac{A\alpha}{2\pi} = a(a+\alpha c)$. Then by lemma 28 we can assume that the immersion x has a helicoidal end at 0 bounded by D_1 and D_2 and of parameters (A,α) (it suffices to consider the good translation in $\mathbb{R}^3$). Moreover we have $\Delta_1^0 \subset D_1$, $\Delta_2 \subset D_2$, Δ_1^0 contains a half of D_1 in the direction of $\vec{u}_1$, and Δ_2 contains a half of D_2 in the direction of $-\vec{u}_2$.

We now prove the necessity of (21). By (20) we have $k_1'k_2 - k_1'k_2 = \lambda^{-1}\mu\alpha F = \varepsilon iF$ with F as in (19). On the other hand, if x has helicoidal ends of parameters (A,α), (B,β) and (C,γ) at 0, ∞ and 1 respectively, then equation (6) holds with φ as in (4), so we get $\varphi = \varepsilon F$, which implies that (21) holds.

From now on we assume that a, b, c and ε satisfy (21). We study the behaviour of x at $z = 1$.

The rotation R defined in section 3.1 moves D_3 onto a horizontal line oriented by the vector $(\cos(\pi\gamma), -\sin(\pi\gamma), 0)$, the vector $\vec{v}_1$ to the vector $-\vec{e}_3$, and it does not change D_2. Let $\tilde{x} = R\circ x$. Let $(\tilde{g},\tilde{\omega})$ be its Weierstrass data, and $\tilde{N}$ its Gauss map. There exists a matrix

$$h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$$

such that

$$\tilde{g} = \frac{h_{22}g + h_{21}}{h_{12}g + h_{11}}, \quad \tilde{\omega} = (h_{12}g + h_{11})^2\omega.$$

Then the associated spinors can be choosen as

$$\tilde{k}_1 = (h_{12}g + h_{11})k_1 = (h_{12}k_2 + h_{11}k_1), \quad \tilde{k}_2 = \tilde{g}\tilde{k}_1 = (h_{22}k_2 + h_{21}k_1).$$

We compute that

$$h=\frac{1}{\sqrt{1+t^2}}\begin{pmatrix}1 & it\\ it & 1\end{pmatrix}.$$

Consequently we have

$$\tilde{k}_1=z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}((a+bz)\tilde{\sigma}_1+cz(1-z)\tilde{\sigma}_1'),$$
$$\tilde{k}_2=z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}((a+bz)\tilde{\sigma}_2+cz(1-z)\tilde{\sigma}_2'),$$

where

$$\begin{pmatrix}\tilde{\sigma}_1\\ \tilde{\sigma}_2\end{pmatrix}=m\begin{pmatrix}w_1^{(1)}\\ w_2^{(1)}\end{pmatrix}$$

with

$$m=\begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}=\frac{1}{\sqrt{1+t^2}}\begin{pmatrix}\lambda^{-1} & \mu it\\ \lambda^{-1}it & \mu\end{pmatrix}\begin{pmatrix}\nu_{11} & \nu_{12}\\ \nu_{21} & \nu_{22}\end{pmatrix}.$$

These expressions are valid for $z\in\Sigma_1$. We notice that $m_{11}\in i\mathbb{R}$ and $m_{22}\in\mathbb{R}$ since $\lambda\in i\mathbb{R}$ and $\mu\in\mathbb{R}$.

We claim that $\tilde{x}$ has a helicoidal end at $z=1$ of parameters (C,γ) bounded by $D_2=R(D_2)$ and $R(D_3)$ if and only if $m_{12}=m_{21}=0$. The proof of this claim is similar to that of lemma 28, so we will only outline the proof. We already know that $\tilde{x}$ maps $(0,1)$ to a part of D_2.

Assume that $\tilde{x}$ has a helicoidal end of parameters (C,γ) bounded by D_2 and $R(D_3)$. Then we must have $g(z)\sim i\rho(1-z)^\gamma$ for some $\rho\in\mathbb{R}^*$. This implies $m_{21}=0$ if $\gamma>0$ and $m_{12}=0$ if $\gamma<0$ (this follows from the fact that $w_1^{(1)}(1)=1$ and $w_2^{(1)}(z)\sim(1-z)^\gamma$ when $z\to1$). Moreover, $\tilde{x}$ maps $(1,2)$ onto a straight line, so the argument of $\overline{\tilde{k}_1^2}-\tilde{k}_2^2$ is constant on $(1,2)$. This implies that $m_{12}=0$ if $m_{21}=0$ and $m_{21}=0$ if $m_{12}=0$ (because $w_1^{(1)}$, $w_2^{(1)}$ and their derivatives take real values on $(1,2)$).

Conversely, we assume that $m_{12}=m_{21}=0$. Since $m_{11}\in i\mathbb{R}$ and $m_{22}\in\mathbb{R}$ we have $\arg\overline{\tilde{k}_1^2}=\arg(-\tilde{k}_2^2)=\pi$ on $(1,0)$ and $\arg\overline{\tilde{k}_1^2}=\arg(-\tilde{k}_2^2)=-\pi\gamma$ on $(1,2)$, and we have $\mathrm{d}(\tilde{x}_1+i\tilde{x}_2)=\overline{\tilde{\omega}}-\tilde{g}^2\tilde{\omega}\sim m_{11}^2(a+b)^2\overline{(1-z)^{-1-\gamma}\mathrm{d}z}-m_{22}^2(a+b-\gamma c)^2(1-z)^{-1+\gamma}\mathrm{d}z$. This proves that $\tilde{x}$ has an end bounded by two lines that are parallel to D_2 and $R(D_3)$, and the signed distance of these lines is equal to $-C$ because of the third equation in (21). Thus, since we know that $\tilde{x}$ maps $(0,1)$ to a part of D_2, we have proved that $\tilde{x}$ has a helicoidal end of parameters (C,γ) bounded by D_2 and $R(D_3)$. This completes proving the claim.

The condition $m_{12}=m_{21}=0$ is satisfied if and only if $\lambda^{-1}\nu_{12}+\mu it\nu_{22}=\lambda^{-1}it\nu_{11}+\mu\nu_{21}=0$, that is, because of (22), if and only if

$$\varepsilon\alpha\mu^2=-\frac{\nu_{12}}{t\nu_{22}}=\frac{t\nu_{11}}{\nu_{21}}. \tag{23}$$

We recall that μ must be real. Thus there exist $\mu\in\mathbb{R}^*$ and $\varepsilon\in\{1,-1\}$ satisfying (23) if and only if $\nu_{12}\nu_{21}=-t^2\nu_{11}\nu_{22}$, which is true by lemma 23.

Henceforth we assume that μ and ε are given by (23). The real number μ is defined uniquely up to its sign, and ε is the sign of $\frac{t\nu_{11}}{\alpha\nu_{21}}$, *i.e.* of $\varepsilon_0\frac{\nu_{11}}{\alpha\nu_{21}}$ by lemma 13. Thus the number λ is also defined uniquely up to its sign by (22). Thus we have proved the necessity of conditions 1 and 2.

We have $\Delta_1^0\subset D_1$, $\Delta_2=D_2$, $\Delta_3^1\subset D_3$, Δ_1^0 contains a half of D_1 in the direction of $\vec{u}_1$ and Δ_3^1 contains a half of D_3 in the direction of $-\vec{u}_3$. It now suffices to check that x has a helicoidal end at ∞ bounded by D_3 and D_1 and of parameters (B,β).

The isometry $T\circ\hat{R}$ defined in section 3.1 moves D_1 onto the x_1-axis oriented by $\vec{e}_1$, D_3 onto a horizontal line oriented by the vector $(\cos(\pi\beta),-\sin(\pi\beta),0)$ and the vector $\vec{v}_\infty$ onto the vector

$-\vec{e}_3$. Let $\hat{x} = T \circ \hat{R} \circ x$. Let $(\hat{g}, \hat{\omega})$ be its Weierstrass data, and $\hat{N}$ its Gauss map. There exists a matrix

$$\hat{h} = \begin{pmatrix} \hat{h}_{11} & \hat{h}_{12} \\ \hat{h}_{21} & \hat{h}_{22} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C})$$

such that

$$\hat{g} = \frac{\hat{h}_{22} g + \hat{h}_{21}}{\hat{h}_{12} g + \hat{h}_{11}}, \quad \hat{\omega} = (\hat{h}_{12} g + \hat{h}_{11})^2 \omega.$$

Then the associated spinors can be choosen as

$$\hat{k}_1 = (\hat{h}_{12} g + \hat{h}_{11}) k_1 = (\hat{h}_{12} k_2 + \hat{h}_{11} k_1), \quad \hat{k}_2 = \hat{g} \hat{k}_1 = (\hat{h}_{22} k_2 + \hat{h}_{21} k_1).$$

We compute that

$$\hat{h} = \frac{1}{\sqrt{1+\hat{t}^2}} \begin{pmatrix} e^{i\pi\frac{\alpha}{2}} & ie^{-i\pi\frac{\alpha}{2}}\hat{t} \\ ie^{i\pi\frac{\alpha}{2}}\hat{t} & e^{-i\pi\frac{\alpha}{2}} \end{pmatrix}.$$

Consequently we have

$$\hat{k}_1 = z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}((a+bz)\hat{\sigma}_1 + cz(1-z)\hat{\sigma}_1'),$$

$$\hat{k}_2 = z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}((a+bz)\hat{\sigma}_2 + cz(1-z)\hat{\sigma}_2'),$$

where

$$\begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \end{pmatrix} = \begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{pmatrix} \begin{pmatrix} w_1^{(\infty)} \\ w_2^{(\infty)} \end{pmatrix}$$

with

$$\begin{pmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{pmatrix} = \frac{1}{\sqrt{1+\hat{t}^2}} \begin{pmatrix} \lambda^{-1} e^{i\pi\frac{\alpha}{2}} & \mu i e^{-i\pi\frac{\alpha}{2}}\hat{t} \\ \lambda^{-1} i e^{i\pi\frac{\alpha}{2}}\hat{t} & \mu e^{-i\pi\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \hat{\nu}_{11} & \hat{\nu}_{12} \\ \hat{\nu}_{21} & \hat{\nu}_{22} \end{pmatrix}.$$

These expressions are valid for $z \in \Sigma_\infty$.

We first prove that $\hat{m}_{12} = \hat{m}_{21} = 0$. Indeed, this condition is equivalent to $\lambda^{-1} e^{i\pi\frac{\alpha}{2}}\hat{\nu}_{12} + \mu i e^{-i\pi\frac{\alpha}{2}}\hat{t}\hat{\nu}_{22} = \lambda^{-1} i e^{i\pi\frac{\alpha}{2}}\hat{t}\hat{\nu}_{11} + \mu e^{-i\pi\frac{\alpha}{2}}\hat{\nu}_{21} = 0$, that is, because of (22), to

$$\varepsilon e^{-i\pi\alpha}\alpha\mu^2 = -\frac{\hat{\nu}_{12}}{\hat{t}\hat{\nu}_{22}} = \frac{\hat{t}\hat{\nu}_{11}}{\hat{\nu}_{21}}. \tag{24}$$

Since μ is defined by (23), this condition is equivalent to

$$\hat{\nu}_{12}\hat{\nu}_{21} = -\hat{t}^2 \hat{\nu}_{11}\hat{\nu}_{22}, \quad \frac{\hat{t}\hat{\nu}_{11}}{\hat{\nu}_{21}} = e^{-i\pi\alpha}\frac{t\nu_{11}}{\nu_{21}},$$

which holds by lemmas 24 and 25. This completes proving that $\hat{m}_{12} = \hat{m}_{21} = 0$.

We claim that $\hat{x}$ has a helicoidal end at $z = \infty$ of parameters (B, β) bounded by $T \circ \hat{R}(D_3)$ and $T \circ \hat{R}(D_1)$. Since $\lambda \in i\mathbb{R}$ and $\mu \in \mathbb{R}$, we have $\arg(\hat{m}_{11}^2) = -\pi(\beta+\gamma)$ and $\arg(\hat{m}_{22}^2) = \pi(1+\beta-\gamma)$, and so $\arg \overline{\hat{k}_1^2} = \arg(-\hat{k}_2^2) = \pi(1+\beta)$ on $(1, +\infty)$ and $\arg \overline{\hat{k}_1^2} = \arg(-\hat{k}_2^2) = 0$ on $(-\infty, -1)$. We also have $\mathrm{d}(\hat{x}_1 + i\hat{x}_2) = \overline{\hat{\omega}} - \hat{g}^2\hat{\omega} \sim \kappa_1 \overline{z^{-1+\beta}\mathrm{d}z} + \kappa_2 z^{-1-\beta}\mathrm{d}z$ for some $\kappa_1, \kappa_2 \in \mathbb{C}^*$. This proves that $\hat{x}$ has an end bounded by two lines that are parallel to $T \circ \hat{R}(D_3)$ and $T \circ \hat{R}(D_3)$, and the signed distance of these lines is equal to $-B$ because of the second equation in (21). Thus, since we know that $\hat{x}$ maps $(1,2)$ to a part of $T \circ \hat{R}(D_3)$, we have proved that $\hat{x}$ has a helicoidal end of parameters (B, β) bounded by $T \circ \hat{R}(D_3)$ and $T \circ \hat{R}(D_3)$. This completes proving the claim, and this also prove that $\Delta_1 = D_1$ and $\Delta_3 = D_3$.

Thus the immersion x is bounded by D_1, D_2 and D_3, and it has helicoidal ends of parameters (A, α), (B, β) and (C, γ) at 0, ∞ and 1 respectively. $\square$

In his memoir [Rie68], Riemann proved the necessity of (21), and only with $\varepsilon = 1$: this comes from the fact that he did not take orientations precisely in consideration. Riemann also noticed the following fact (page 335).

Lemma 32 (Riemann, [Rie68])**.** *System (21) is equivalent to*

$$\left\{ \begin{array}{rcl} p^2 - \alpha^2(p+q+r)^2 & = & \varepsilon\frac{A\alpha}{2\pi} \\ q^2 - \beta^2(p+q+r)^2 & = & \varepsilon\frac{B\beta}{2\pi} \\ r^2 - \gamma^2(p+q+r)^2 & = & \varepsilon\frac{C\gamma}{2\pi} \end{array} \right. \tag{25}$$

with $p = a + \frac{\alpha}{2}c$, $q = b + \frac{1-\alpha-\gamma}{2}c$ *and* $r = -a - b + \frac{\gamma}{2}c$.

Theorem 33. *A real solution (p, q, r) of system (25), where ε is the sign of $\varepsilon_0 \frac{\nu_{11}}{\alpha\nu_{21}}$, gives a minimal immersion $x : \Sigma \to \mathbb{R}^3$ (possibly with a singular point at a double root of φ defined by (4)) bounded by (D_1, D_2, D_3) and having helicoidal ends of parameters (A, α), (B, β) and (C, γ) at 0, ∞ and 1 respectively. We will denote it $\mathcal{I}(\alpha, \gamma, \beta, \varepsilon_0, p, r, q)$.*

Moreover, two different real solutions of system (25) give the same immersion if and only if they are opposite one to the other.

Proof. The first assertion is a consequence of proposition 28 and lemma 32.

Assume that (p, q, r) and $(\hat{p}, \hat{q}, \hat{r})$ are two real solution of (25). Then they correspond to two solutions (a, b, c) and $(\hat{a}, \hat{b}, \hat{c})$ of (21), which define functions K_1, K_2, $\hat{K}_1$, $\hat{K}_2$ by (18). If they define the same immersion, then their Weierstrass data are equal, and so $\hat{K}_1 = \pm K_1$ and $\hat{K}_2 = \pm K_2$ (because λ and μ are determined), which implies $(\hat{a}, \hat{b}, \hat{c}) = \pm(a, b, c)$, and finally $(\hat{p}, \hat{q}, \hat{r}) = \pm(p, q, r)$. The converse is clear. □

Corollary 34. *If $\alpha, \beta, \gamma \in (0, 1)$ (that is if $\alpha = \alpha_0$, $\beta = \beta_0$ and $\gamma = \gamma_0$), then there exists a minimal immersion $x : \Sigma \to \mathbb{R}^3$ (possibly with a singular point at a double root of φ) bounded by (D_1, D_2, D_3) and having helicoidal ends of parameters (A, α), (B, β) and (C, γ) at 0, ∞ and 1 respectively.*

Proof. It suffices to prove that system (25) has at least one real solution. We set

$$p(y) = \sqrt{\varepsilon\frac{A\alpha}{2\pi} + \alpha^2 y^2},$$

$$q(y) = \sqrt{\varepsilon\frac{B\beta}{2\pi} + \beta^2 y^2},$$

$$r(y) = \sqrt{\varepsilon\frac{C\gamma}{2\pi} + \gamma^2 y^2},$$

and we define eight functions

$$F_{\pm\pm\pm}(y) = \pm p(y) \pm q(y) \pm r(y) - y$$

for real numbers y such that these expressions are defined. Then system (25) has a real solution if and only if at least one of the eight equations $F_{\pm\pm\pm}(y) = 0$ has a solution $y \in \mathbb{R}$. When $y \to +\infty$, we have $F_{\pm\pm\pm}(y) \sim (\pm\alpha \pm \beta \pm \gamma - 1)y$. When $y \to -\infty$, we have $F_{\pm\pm\pm}(y) \sim (\mp\alpha \mp \beta \mp \gamma - 1)y$.

We first deal with the case where $\varepsilon A\alpha$, $\varepsilon B\beta$ and $\varepsilon C\gamma$ are all positive. Then the functions $F_{\pm\pm\pm}$ are defined on the whole $\mathbb{R}$. Moreover we have $\alpha + \beta - \gamma - 1 < 0$ and $-\alpha - \beta + \gamma - 1 < 0$ because $(\alpha, \beta, \gamma) \in \mathcal{K}$. Thus by the intermediate value theorem equation $F_{++-}(y) = 0$ has a solution $y \in \mathbb{R}$.

We now deal with the case where at least one of the numbers $\varepsilon A\alpha$, $\varepsilon B\beta$ and $\varepsilon C\gamma$ is negative (for example $\varepsilon B\beta$). We can assume that $\frac{\varepsilon B}{2\pi\beta} \leqslant \frac{\varepsilon A}{2\pi\alpha}$ and $\frac{\varepsilon B}{2\pi\beta} \leqslant \frac{\varepsilon C}{2\pi\gamma}$. Then the functions $F_{\pm,\pm,\pm}$

are defined for $|y| \geqslant y_0$ where $y_0 = \sqrt{\frac{|B|}{2\pi\beta}}$. We have $F_{+++}(y_0) = F_{+-+}(y_0)$, $\alpha+\beta+\gamma-1>0$ and $\alpha-\beta+\gamma-1<0$ (because $(\alpha,\beta,\gamma)\in\mathcal{K}$). Thus by the intermediate value theorem there exists $y\in\mathbb{R}$ such that $F_{+++}(y)=0$ or $F_{+-+}(y)=0$ (depending on the sign of $F_{+++}(y_0)$). ☐

Remark 35. *If $\alpha,\beta,\gamma\in(0,1)$, then we have $\varepsilon=\varepsilon_0$.*

3.5. **The differential equation satisfied by K_1 and K_2.** This section contains technical results that will be used to construct trinoids in hyperbolic space (section 4). We assume here that the numbers p, q and r defined in lemma 32 satisfy system (25), and that they are real.

Lemma 36. *Set $\hat{\Theta}\mathrm{d}z^2 = \mathrm{S}_z h - \mathrm{S}_z\zeta$ where $h=\frac{K_2}{K_1}$ and $\zeta(z)=\int_0^z\varphi(\tau)\mathrm{d}\tau$. Then K_1 and K_2 are solutions on Σ of the following differential equation:*

$$K'' - \frac{\varphi'}{\varphi}K' + \frac{\hat{\Theta}}{2}K = 0. \tag{26}$$

Proof. We know that K_1 and K_2 satisfy (20) with $F=\varepsilon\varphi$ (because (p,q,r) is a solution of (25)). Then the proof is the same as that of equation (7) in lemma 17. ☐

Lemma 37. *The function $\hat{\Theta}$ extends to a meromorphic function on $\bar{\mathbb{C}}$. Moreover we have*

$$\hat{\Theta} = \frac{\Phi}{z^2(z-1)^2} + \frac{\hat{\Lambda}}{z(z-1)\varphi} + \frac{2\varphi''}{\varphi}$$

with Φ as in (8) and wheer $\hat{\Lambda}$ is an affine function.

Proof. By proposition 31 there exist $\lambda,\mu\in\mathbb{C}^*$ such that the functions $k_1=\lambda^{-1}K_1$ and $k_2=\mu K_2$ define a minimal immersion bounded by some $(D_1,D_2,D_3)\in\mathcal{D}$, with possibly a singular point if φ has a double root). Then the Gauss map of this immersion is $g=\lambda\mu h$, so $\mathrm{S}_z h=\mathrm{S}_z g$. Thus $\hat{\Theta}$ extends to a meromorphic function on $\bar{\mathbb{C}}$ by Schwarz reflection, and the expression of $\hat{\Theta}$ follows from lemma 18 (this remains true in the case where φ has a double root, since the order of the pole of $\mathrm{S}_z h$ at the root of φ is at most 2). ☐

Lemma 38. *We have*

$$\hat{\Lambda}(0) = \varepsilon(p+q+r)((\gamma^2-\beta^2)(2p+q+r)+(1-\alpha^2)(q-r)),$$

$$\hat{\Lambda}(1) = \varepsilon(p+q+r)((\alpha^2-\beta^2)(p+q+2r)+(1-\gamma^2)(q-p)),$$

$$\hat{\Lambda}(1)-\hat{\Lambda}(0) = \varepsilon(p+q+r)((\alpha^2-\gamma^2)(p+2q+r)+(1-\beta^2)(r-p)).$$

Proof. We recall that in the neighbourhood of 0 we have

$$K_j = z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}\left((a+bz)w_j^{(0)} + cz(1-z)(w_j^{(0)})'\right)$$

for $j=1,2$, with $w_j^{(0)}$ as in section 3.3. We have

$$h(z) = z^\alpha\frac{a+\alpha c+\left(b-\alpha c+\frac{s_{+--}-s_{++-}}{1+\alpha}(a+(1+\alpha)c)\right)z+\mathrm{O}(z^2)}{a+\left(b+\frac{s_{---}-s_{-+-}}{1-\alpha}(a+c)\right)z+\mathrm{O}(z^2)}.$$

The coefficient of the order -1 term in $\hat{\Theta}$ at 0 is $\hat{s}_{-1} = \frac{1-\alpha^2}{\alpha}\frac{h_1}{h_0}$ where $h(z) = z^\alpha(h_0+h_1z+\mathrm{O}(z^2))$. We compute that

$$\begin{aligned}\hat{s}_{-1} &= \frac{1}{2(p^2-\alpha^2(p+q+r)^2)} \times \\ &(\alpha^2(\alpha^2-\beta^2+\gamma^2-1)(p+q+r)^2 + (-\alpha^2+5\beta^2-5\gamma^2+1)p^2 \\ &+(2\alpha^2+2\beta^2-2\gamma^2-2)q^2 + (-2\alpha^2+2\beta^2-2\gamma^2+2)r^2 \\ &+2(\alpha^2+3\beta^2-3\gamma^2-1)pq + 2(-\alpha^2+3\beta^2-3\gamma^2+1)pr \\ &+4(\beta^2-\gamma^2)qr).\end{aligned}$$

Using that $p^2-\alpha^2(p+q+r)^2 = \varepsilon\frac{A\alpha}{2\pi}$ we get that

$$\begin{aligned}\hat{s}_{-1} &= \varepsilon\frac{\pi}{A\alpha} \times \\ &(-(\alpha^2-\beta^2+\gamma^2-1)\varepsilon\frac{A\alpha}{2\pi} + 4(\beta^2-\gamma^2)p^2 \\ &+(2\alpha^2+2\beta^2-2\gamma^2-2)q^2 + (-2\alpha^2+2\beta^2-2\gamma^2+2)r^2 \\ &+2(\alpha^2+3\beta^2-3\gamma^2-1)pq + 2(-\alpha^2+3\beta^2-3\gamma^2+1)pr \\ &+4(\beta^2-\gamma^2)qr).\end{aligned}$$

On the other hand we have $\hat{s}_{-1} = -\frac{\hat{\Lambda}(0)}{\varphi(0)} + \Phi'(0) + 2\Phi(0)$, so using the fact that $\Phi'(0)+2\Phi(0) = \frac{1-\alpha^2+\beta^2-\gamma^2}{2}$ and $\varphi(0) = \frac{A\alpha}{2\pi}$ we conclude that

$$\hat{\Lambda}(0) = \varepsilon(p+q+r)((\gamma^2-\beta^2)(2p+q+r) + (1-\alpha^2)(q-r)).$$

In the neighbourhood of 1 we have

$$K_j = z^{\frac{1-\alpha}{2}}(1-z)^{\frac{1-\gamma}{2}}\left((a+bz)w_j^{(1)} + cz(1-z)(w_j^{(1)})'\right)$$

for $j=1,2$, with $w_j^{(1)}$ as in section 3.3. In the same way we compute that

$$\hat{\Lambda}(1) = \varepsilon(p+q+r)((\alpha^2-\beta^2)(p+q+2r) + (1-\gamma^2)(q-p)),$$

and we deduce the expression of $\hat{\Lambda}(1)-\hat{\Lambda}(0)$. □

4. Application to trinoids in hyperbolic space

4.1. **The cousin relation.** We recall a few facts about the cousin relation between minimal surfaces in $\mathbb{R}^3$ and Bryant surfaces, *i.e.* constant mean curvature one (CMC-1) surfaces in hyperbolic space $\mathbb{H}^3$. The asymptotic boundary of $\mathbb{H}^3$ will be denoted by $\partial_\infty\mathbb{H}^3$.

Let $\mathcal{S}$ be a simply connected Riemann surface. If $x:\mathcal{S}\to\mathbb{R}^3$ is a conformal minimal immersion, I and II its first and second fundamental forms, then there exists a conformal CMC-1 immersion $\tilde{x}:\mathcal{S}\to\mathbb{H}^3$ whose first and second fundamental forms are

$$\tilde{\mathrm{I}} = \mathrm{I}, \quad \widetilde{\mathrm{II}} = \mathrm{II} + \mathrm{I},$$

and conversely. The immersions x and $\tilde{x}$ are said to be cousin immersions. They are unique up to isometries of $\mathbb{R}^3$ and $\mathbb{H}^3$ respectively.

Bryant proved in [Bry87] that if $\tilde{x}$ is such an immersion (with $\mathcal{S}$ non necessarily simply connected), then there exists a holomorphic immersion $F:\tilde{\mathcal{S}}\to\mathrm{SL}_2(\mathbb{C})$ where $\tilde{\mathcal{S}}$ is the universal cover of $\mathcal{S}$ such that $\tilde{x} = FF^*$ and $\det(F^{-1}\mathrm{d}F) = 0$, where the model of hyperbolic space is

$$\mathbb{H}^3 = \{M\in\mathcal{M}_2(\mathbb{C}) | M^* = M, \operatorname{tr} M > 0, \det M = 1\}.$$

Moreover we have

$$F^{-1}\mathrm{d}F = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega$$

where (g, ω) are the Weierstrass data of the cousin immersion x (see also [UY93]). The map F is called the Bryant representation of x.

The geodesic lines of curvature of x correspond to the geodesic lines of curvature of $\tilde{x}$, and they lie in planes that are orthogonal to the surface. The Schwarz reflexion principle for geodesic lines of curvature also holds for Bryant surfaces. Thus a planar symmetry of x correspond to a planar symmetry of $\tilde{x}$. These facts are explained in details in [Kar01] and [SET01].

The cousin immersion of the conjugate immersion of x will be called the conjugate cousin immersion of x and it will be denoted by x°. Thus the Weierstrass data of x° are $(g^\circ, \omega^\circ) = (g, i\omega)$. Moreover, straight lines of x correspond to geodesic lines of curvature of x° (hence lying in hyperbolic planes), and symmetries of x with respect to a straight line correspond to symmetries of x° with respect to a hyperbolic plane.

4.2. **Trinoids.** Sá Earp and Toubiana proved in [SET01] that an embedded end of finite total curvature is either asymptotic to the end of a rotational catenoid cousin (in which case it is called a catenoidal end) or to a horosphere (in which case it is called a horospherical end). The catenoidal ends are the embedded type I ends in the sense of [UY93]; they are asymptotically rotational surfaces (see [Dan02]).

Definition 39. *A Bryant surface is called a trinoid if it has genus zero and three catenoidal ends.*

Consequently, a trinoid is given by a conformal CMC-1 immersion defined on $\mathbb{C} \setminus \{0, 1\}$. The three ends correspond to 0, 1 and ∞.

The definitions of a catenoidal end and of a trinoid in [BPS02] are slightly different from ours. However it turns out that the definitions of trinoids are equivalent.

In [CHR01], Collin, Hauswirth and Rosenberg proved many results about properly embedded Bryant surfaces: a properly embedded Bryant surface of genus 0 with 1 end (respectively 2 ends, 3 ends) is a horosphere (respectively an embedded catenoid cousin, an embedded trinoid).

The aim of this section is to construct trinoids by the method of the conjugate cousin immersion. We first prove the following proposition, which is a reformulation of lemma 2.4 in [SET01] and which will be useful in the sequel.

Proposition 40. *Let $\mathcal{O}$ be a neighbourhood of* 0 *in* $\mathbb{C}$, $\mathcal{O}^* = \mathcal{O} \setminus \{0\}$ *and* $\widetilde{\mathcal{O}^*}$ *be the universal cover of* $\mathcal{O}^*$. *Let* $\mu \in \mathbb{R}^*$ *and let* $x : \widetilde{\mathcal{O}^*} \to \mathbb{H}^3$ *be a conformal CMC-*1 *immersion whose Weierstrass data* (g, ω) *satisfy*

$$g(z) \sim g_0 z^\mu, \quad \omega \sim \omega_0 z^{-1-\mu} \mathrm{d}z$$

when $z \to 0$ *with* $g_0, \omega_0 \in \mathbb{C}^*$. *Let* Q *be its Hopf differential.*

Then x *is an embedding of a punctured neighbourhood of* 0 *in* $\mathcal{O}^*$ *if and only if the* 2*-form* $\mathrm{S}_z g - 2Q$ *is holomorphic at* 0.

Proof. We set

$$g(z) = z^\mu (g_0 + g_1 z + \mathrm{O}(z^2)), \quad \omega = z^{-1-\mu}(\omega_0 + \omega_1 z + \mathrm{O}(z^2))\mathrm{d}z.$$

Then we have

$$Q = z^{-2}(q_{-2} + q_{-1} z + \mathrm{O}(1))\mathrm{d}z^2, \quad \mathrm{S}_z g = z^{-2}(s_{-2} + s_{-1} z + \mathrm{O}(1))\mathrm{d}z^2$$

with $q_{-2} = \mu\omega_0 g_0$, $q_{-1} = \mu\omega_1 g_0 + (1+\mu)\omega_0 g_1$, $s_{-2} = \frac{1-\mu^2}{2}$, $s_{-1} = \frac{1-\mu^2}{\mu}\frac{g_1}{g_0}$.

We first assume that $\mu > 0$. Then, according to lemma 2.4 in [SET01], x is an embedding of a punctured neighbourhood of 0 in $\mathcal{O}^*$ if and only if

$$g_0\omega_0 = \frac{1-\mu^2}{4\mu}, \quad (1+\mu)\frac{\omega_1}{\omega_0} = 2\mu\omega_1 g_0 + 2(1+\mu)\omega_0 g_1. \tag{27}$$

The first condition in (27) is equivalent to $s_{-2} = 2q_{-2}$ (this means that $\mathrm{S}_z g - 2Q$ has at most a pole of order 1 at 0). If this condition is satisfied, then since $\frac{q_{-1}}{q_{-2}} = \frac{\omega_1}{\omega_0} + \frac{1+\mu}{\mu}\frac{g_1}{g_0}$, the second condition in (27) is equivalent to $(1+\mu)\left(\frac{q_{-1}}{q_{-2}} - \frac{s_{-1}}{1-\mu}\right) = 2q_{-1}$, and thus to $s_{-1} = 2q_{-1}$, which completes the proof in the case where $\mu > 0$.

We now deal with the case where $\mu < 0$. The data $(g^{-1}, -g^2\omega)$ define the same CMC-1 immersion as (g,ω) (see for example [UY93]), and we have $g^{-1}(z) \sim g_0^{-1}z^{-\mu}$, $-g^2\omega \sim -g_0^2\omega_0 z^{-1+\mu}\mathrm{d}z$, and $\mathrm{S}_z g - 2Q$ is unchanged, so it suffices to apply the previous case with $-\mu$ instead of μ. □

Remark 41. *Umehara and Yamada proved in* [UY93] *that* $\mathrm{S}_z G = \mathrm{S}_z g - 2Q$ *where* G *is the hyperbolic Gauss map of the immersion* x.

Let $(D_1, D_2, D_3) \in \mathcal{D}$, $(\alpha_0, \gamma_0, \beta_0, -A, -C, -B, \varepsilon_0) = L(D_1, D_2, D_3)$ (see section 3.1), $\alpha \in \alpha_0 + 2\mathbb{Z}$, $\beta \in \beta_0 + 2\mathbb{Z}$, $\gamma \in \gamma_0 + 2\mathbb{Z}$. Let $x : \Sigma \to \mathbb{R}^3$ be a minimal immersion bounded by (D_1, D_2, D_3) or its dual configuration, and having helicoidal ends of parameters (A,α), (B,β) and (C,γ) at 0, 1 and ∞ respectively, corresponding to a solution (p,q,r) of (25) where $\varepsilon = \pm 1$. Let (g,ω) be its Weierstrass data and $Q = \omega\mathrm{d}g$ its Hopf differential.

Then the conjugate cousin immersion $x^\circ : \Sigma \to \mathbb{H}^3$ maps $(-\infty, 0)$, $(0,1)$ and $(1,\infty)$ onto three geodesic lines of curvature $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ belonging to three hyperbolic planes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$.

Proposition 42. *If* $\frac{\alpha^2}{4} > \frac{A\alpha}{2\pi}$, $\frac{\beta^2}{4} > \frac{B\beta}{2\pi}$ *and* $\frac{\gamma^2}{4} > \frac{C\gamma}{2\pi}$, *then the asymptotic boundary of each end of* x° *consists of one point.*

Proof. It suffices to prove that the asymptotic boundary of the end at 0 consists of one point. The Hopf differential $Q^\circ = \omega^\circ\mathrm{d}g^\circ$ satisfies $Q^\circ \sim q_{-2}z^{-2}\mathrm{d}z$ at 0 with $q_{-2} = -\frac{A\alpha}{2\pi}$. We proceeding as in the proof of lemma 2.4 in [SET01]: since the indicial equations $\tau^2 + \alpha\tau - q_{-2}$ and $\upsilon^2 - \alpha\upsilon - q_{-2}$ have a positive discriminant $\Delta = \alpha^2 + 4q_{-2}$ (because of the hypothesis), we prove that, up to an isometry of $\mathbb{H}^3$, the Bryant representation of x is $F = \begin{pmatrix} z^\tau A_1(z) & z^\upsilon B_1(z) \\ z^\tau C_1(z) & z^\upsilon D_1(z) \end{pmatrix}$ where $\tau = \frac{-\sqrt{\Delta}-\alpha}{2}$, $\upsilon = \frac{-\sqrt{\Delta}+\alpha}{2}$ and where A_1, B_1, C_1 and D_1 are holomorphic functions in a neighbourhood of 0 in $\{\operatorname{Im} z \geqslant 0\}$ that do not vanish at 0. Using the identification of $\mathbb{H}^3$ with the upper half-space model $\{(y_1,y_2,y_3) \in \mathbb{R}^3 | y_3 > 0\}$ described in [SET01], we get

$$y_1 + iy_2 = \frac{|z|^{2\tau}\overline{A_1}C_1 + |z|^{2\upsilon}\overline{B_1}D_1}{|z|^{2\tau}|A_1|^2 + |z|^{2\upsilon}|B_1|^2}, \quad y_3 = \frac{1}{|z|^{2\tau}|A_1|^2 + |z|^{2\upsilon}|B_1|^2}$$

(this is formula (1.2) of [SET01]). Thus we have $y_3 \to 0$ when $z \to 0$ (since τ or υ is negative), and $y_1 + iy_2 \to \frac{C_1(0)}{A_1(0)}$ or $y_1 + iy_2 \to \frac{D_1(0)}{B_1(0)}$ (depending on the sign of α). This proves the assertion. □

From now on we assume that $\frac{\alpha^2}{4} > \frac{A\alpha}{2\pi}$, $\frac{\beta^2}{4} > \frac{B\beta}{2\pi}$ and $\frac{\gamma^2}{4} > \frac{C\gamma}{2\pi}$. Thus the lines $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are pairwise concurrent in $\partial_\infty\mathbb{H}^3$ at the asymptotic boundary points. Applying Schwarz reflections with respect to these planes and repeating the process with respect to the new planes infinitely many times, we get a conformal CMC-1 immersion $x^\circ : \widetilde{\mathbb{C}\setminus\{0,1\}} \to \mathbb{H}^3$ where $\widetilde{\mathbb{C}\setminus\{0,1\}}$ is the universal cover of $\mathbb{C}\setminus\{0,1\}$. This immersion x° is well-defined on $\mathbb{C}\setminus\{0,1\}$ if and only if the planes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are equal.

Proposition 43. *If immersion x° gives a trinoid by Schwarz reflection, then we have*

$$\frac{A\alpha}{2\pi} = \frac{\alpha^2-1}{4}, \quad \frac{B\beta}{2\pi} = \frac{\beta^2-1}{4}, \quad \frac{C\gamma}{2\pi} = \frac{\gamma^2-1}{4}. \tag{28}$$

Proof. Assume that x° gives a trinoid by Schwarz reflection. Then x° is well-defined on $\mathbb{C}\backslash\{0,1\}$, and its ends are embedded. Let Q° be its Hopf differential. We have $Q^\circ = iQ$.

Its Weierstrass data satisfy $g^\circ(z) = g(z) \sim g_0 z^\alpha$ and $\omega^\circ = i\omega \sim \omega_0 z^{-1-\alpha}\mathrm{d}z$ when $z \to 0$, with $g_0, \omega_0 \in \mathbb{C}^*$. Then by proposition 40 the form $\mathrm{S}_z g^\circ - 2Q^\circ$ is holomorphic at 0. In particular the order -2 term vanishes, that is $\frac{1-\alpha^2}{2} + 2\frac{A\alpha}{2\pi} = 0$.

The other two identities are obtained in the same way for the ends at ∞ and 1. □

A computation gives the following result.

Lemma 44. *The complex solutions of system*

$$\left\{\begin{array}{rcl} p^2-\alpha^2(p+q+r)^2 &=& \frac{\alpha^2-1}{4} \\ q^2-\beta^2(p+q+r)^2 &=& \frac{\beta^2-1}{4} \\ r^2-\gamma^2(p+q+r)^2 &=& \frac{\gamma^2-1}{4} \end{array}\right. \tag{29}$$

are $(-\frac{i}{2},\frac{i}{2},\frac{i}{2})$, $(\frac{i}{2},-\frac{i}{2},\frac{i}{2})$, $(\frac{i}{2},\frac{i}{2},-\frac{i}{2})$, $(U\delta, V\delta, W\delta)$ and their opposites, where

$$\begin{aligned} U &= -3\alpha^4 + 2(1+\beta^2+\gamma^2)\alpha^2 + \beta^4 + \gamma^4 - 2(\beta^2+\gamma^2+\beta^2\gamma^2) + 1, \\ V &= -3\beta^4 + 2(1+\alpha^2+\gamma^2)\beta^2 + \alpha^4 + \gamma^4 - 2(\alpha^2+\gamma^2+\alpha^2\gamma^2) + 1, \\ W &= -3\gamma^4 + 2(1+\alpha^2+\beta^2)\gamma^2 + \alpha^4 + \beta^4 - 2(\alpha^2+\beta^2+\alpha^2\beta^2) + 1, \end{aligned}$$

and where δ is a complex square root of $-\frac{1}{4\Pi}$ with Π defined by (13).

The complex solutions of system

$$\left\{\begin{array}{rcl} p^2-\alpha^2(p+q+r)^2 &=& \frac{1-\alpha^2}{4} \\ q^2-\beta^2(p+q+r)^2 &=& \frac{1-\beta^2}{4} \\ r^2-\gamma^2(p+q+r)^2 &=& \frac{1-\gamma^2}{4} \end{array}\right. \tag{30}$$

are $(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2},\frac{1}{2})$, $(-\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $(iU\delta, iV\delta, iW\delta)$ and their opposites.

Remark 45. *These solutions are distinct if and only if $1-\alpha^2-\beta^2+\gamma^2 \neq 0$, $1-\alpha^2+\beta^2-\gamma^2 \neq 0$ and $1+\alpha^2-\beta^2-\gamma^2 \neq 0$.*

Remark 46. *We compute that $U+V+W$ is -4 times the discriminant of Φ.*

Proposition 47. *The immersion x° gives a trinoid by Schwarz reflection if and only if (28) holds, φ has no double root, and $(p,q,r) = \pm(U\delta, V\delta, W\delta)$ in the case where $\delta \in \mathbb{R}$ or $(p,q,r) = \pm(iU\delta, iV\delta, iW\delta)$ in the case where $\delta \in i\mathbb{R}$, where U, V, W and δ are as in lemma 44.*

Proof. We have

$$\mathrm{S}_z g^\circ - 2Q^\circ = \left(\frac{\Phi+2\varphi}{z^2(z-1)^2} + \frac{\Lambda}{\varphi z(z-1)} + \frac{2\varphi''}{\varphi}\right)\mathrm{d}z^2 + \mathrm{S}_z\zeta$$

with $\zeta(z) = \int_0^z \varphi(\tau)\mathrm{d}\tau$, Φ as in (8) and Λ as $\hat{\Lambda}$ in lemma 38. By proposition 40, x° gives a trinoid if and only if $\mathrm{S}_z g^\circ - Q^\circ$ is holomorphic at 0, 1 and ∞. This holds if and only if $\Phi = -2\varphi$ (*i.e.* if (28) holds, *i.e.* if (p,q,r) is a real solution of (29) or (30)) and $\Lambda = 0$ (we recall that $\frac{2\varphi''}{\varphi}\mathrm{d}z^2 + \mathrm{S}_z\zeta$ is holomorphic at ∞).

By lemma 38 we have $\Lambda = 0$ if and only if $p+q+r = 0$ or

$$\left\{\begin{array}{rcl} (\gamma^2-\beta^2)(2p+q+r) + (1-\alpha^2)(q-r) &=& 0 \\ (\alpha^2-\beta^2)(p+q+2r) + (1-\gamma^2)(q-p) &=& 0. \end{array}\right. \tag{31}$$

We notice that $(U,V,W)\neq(0,0,0)$ (since $(U\delta,V\delta,W\delta)$ is solution of (29)). Thus the set of the solutions of (31) is the complex line $\mathbb{C}(U,V,W)$. Hence the only solutions of (31) that are also solutions of (29) or (30) are $(U\delta,V\delta,W\delta)$, $(iU\delta,iV\delta,iW\delta)$ and their opposites.

There is a solution (p,q,r) of (29) or (30) satisfying $p+q+r=0$ if and only if $U+V+W=0$, *i.e.* if and only if φ has a double real root a_1 (by remark 46 and since $\Phi=-2\varphi$). In this case these solutions are again $(U\delta,V\delta,W\delta)$, $(iU\delta,iV\delta,iW\delta)$ and their opposites.

Hence, in the case where φ has a double root a_1, the solutions (p,q,r) satisfy $p+q+r=0$, which implies $\Lambda=0$ by lemma 38. Thus the exponents of equation (7) at a_1 are 1 and 2. This implies that the spinors k_1 and k_2 associated to x both vanish at a_1, and so x and x° do have a singular point at a_1.

Moreover, the numbers p, q and r are required to be real, so the proof is complete. □

Remark 48. *In the case where φ has a double root, the result still holds except that the immersion giving the trinoid has a singularity at the root of φ.*

Theorem 49. *Let μ_0, μ_1 and μ_∞ be three positive non-integer real numbers. Assume that*

$$(|[\mu_0]|,|[\mu_1]|,|[\mu_\infty]|)\in\mathcal{K},\tag{32}$$

where $[r]$ denotes the unique number in $(-1,1]$ such that $r-[r]\in2\mathbb{Z}$ (the set $\mathcal{K}$ is defined in proposition 9), and that

$$\mu_0^4+\mu_1^4+\mu_\infty^4-2\mu_0^2\mu_1^2-2\mu_0^2\mu_\infty^2-2\mu_1^2\mu_\infty^2+2\mu_0^2+2\mu_1^2+2\mu_\infty^2-3\neq0.\tag{33}$$

Then there exists a trinoid $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ whose ends are of growths $1-\mu_0$, $1-\mu_1$ and $1-\mu_\infty$ and having a symmetry plane.

The ends of $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ have distinct asymptotic boundary points if and only if $1-\mu_0^2-\mu_1^2+\mu_\infty^2\neq0$, $1-\mu_0^2+\mu_1^2-\mu_\infty^2\neq0$ and $1+\mu_0^2-\mu_1^2-\mu_\infty^2\neq0$. More precisely, the ends of growths $1-\mu_0$ and $1-\mu_1$ (respectively $1-\mu_0$ and $1-\mu_\infty$, $1-\mu_1$ and $1-\mu_\infty$) have distinct asymptotic boundary points if and only if $1-\mu_0^2-\mu_1^2+\mu_\infty^2\neq0$ (respectively $1-\mu_0^2+\mu_1^2-\mu_\infty^2\neq0$, $1+\mu_0^2-\mu_1^2-\mu_\infty^2\neq0$). In particular the three ends cannot have the same asymptotic boundary point.

Proof. We set $\alpha=\mu_0$ if $\mu_0\in|[\mu_0]|+2\mathbb{Z}$ and $\alpha=-\mu_0$ if $-\mu_0\in|[\mu_0]|+2\mathbb{Z}$ (in order to be compatible with the conventions of section 3.2). In the same way we set $\beta=\pm\mu_\infty$ and $\gamma=\pm\mu_1$. Let $\varepsilon_0\in\{1,-1\}$. By proposition 9 there exists a triple (D_1,D_2,D_3) such that $L(D_1,D_2,D_3)=(|[\mu_0]|,|[\mu_1]|,|[\mu_\infty]|,-A,-C,-B,\varepsilon_0)$ with $\frac{A\alpha}{2\pi}=\frac{\alpha^2-1}{4}$, $\frac{B\beta}{2\pi}=\frac{\beta^2-1}{4}$, $\frac{C\gamma}{2\pi}=\frac{\gamma^2-1}{4}$. Then by (33) the corresponding φ has no double root, and so by proposition 47 there exists a minimal immersion $x:\Sigma\to\mathbb{R}^3$ bounded by (D_1,D_2,D_3) or its dual configuration whose conjugate cousin x° gives a trinoid by Schwarz reflection ; moreover the growths of the ends of this trinoid are $1-\mu_0$, $1-\mu_1$ and $1-\mu_\infty$ respectively (it suffices to consider the coefficient of the order -2 term of Q° at each end). This proves the existence of the trinoid $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ (it has a symmetry plane by construction).

Up to an isometry of $\mathbb{H}^3$, the hyperbolic Gauss map of x° is $G^\circ(z)=z+\frac{(a_1-a_2)^2}{2(2z-a_1-a_2)}$ where a_1 and a_2 are the roots of φ (see [RUY01], example 4.4; we notice that $\mathrm{S}_zG^\circ=\mathrm{S}_z\zeta+\frac{2\varphi''}{\varphi}\mathrm{d}z^2$). Moreover, the limit of the hyperbolic Gauss map at a catenoidal end is the asymptotic boundary point of the end (see [SET01]). Thus, to compare the asymptotic boundary points of the ends, it suffices to compare $G^\circ(0)$, $G^\circ(1)$ and $G^\circ(\infty)$.

We have $G^\circ(0)=-\frac{(a_1-a_2)^2}{2(a_1+a_2)}$, $G^\circ(1)=1+\frac{(a_1-a_2)^2}{2(2-a_1-a_2)}$ and $G^\circ(\infty)=\infty$. Thus we have $G^\circ(0)=G^\circ(1)$ if and only if $a_1+a_2=2a_1a_2$, *i.e.* $1-\mu_0^2-\mu_1^2+\mu_\infty^2=0$; we have $G^\circ(0)=G^\circ(\infty)$ if and only if $a_1+a_2=0$, *i.e.* $1-\mu_0^2+\mu_1^2-\mu_\infty^2=0$; we have $G^\circ(1)=G^\circ(\infty)$ if and only if $a_1+a_2=2$, *i.e.* $1+\mu_0^2-\mu_1^2-\mu_\infty^2=0$. This also implies that we never have $G^\circ(0)=G^\circ(1)=G^\circ(\infty)$. □

Remark 50. *If (32) holds but (33) does not hold, then there exists a "trinoid" $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ with one singular point.*

Corollary 51. *If $(\mu_0, \mu_1, \mu_\infty) \in \mathcal{K}$ and $\mu_0, \mu_1, \mu_\infty \in (0, 1)$ (i.e. if the growths are positive), then the trinoid $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ exists and its ends have distinct asymptotic boundary points.*

Proof. In this case we have $\mu_j = |[\mu_j]|$ for $j = 0, 1, \infty$, and so $(\mu_0, \mu_1, \mu_\infty) \in \mathcal{K}$. This implies that $\mu_\infty > 1-\mu_0-\mu_1$ and $\mu_\infty > -1+\mu_0+\mu_1$, and so $1-\mu_0^2-\mu_1^2+\mu_\infty^2 > 1-\mu_0^2-\mu_1^2+(1-\mu_0-\mu_1)^2 = 2(1-\mu_0)(1-\mu_1) > 0$. In the same way we have $1-\mu_0^2+\mu_1^2-\mu_\infty^2 \neq 0$ and $1+\mu_0^2-\mu_1^2-\mu_\infty^2 \neq 0$. Thus the asymptotic boundary points of the ends are distinct.

Set $d(\mu_0, \mu_1, \mu_\infty) = \mu_0^4 + \mu_1^4 + \mu_\infty^4 - 2\mu_0^2\mu_1^2 - 2\mu_0^2\mu_\infty^2 - 2\mu_1^2\mu_\infty^2 + 2\mu_0^2 + 2\mu_1^2 + 2\mu_\infty^2 - 3$. The derivative of d with respect to μ_∞^2 is equal to $2(\mu_\infty^2 - \mu_0^2 - \mu_1^2 + 1)$, which was proven to be positive. Without loss of generality we can assume that $\mu_0 \geqslant \mu_1$. We have $\mu_\infty < 1 - \mu_0 + \mu_1$, and so $d(\mu_0, \mu_1, \mu_\infty) < d(\mu_0, \mu_1, 1 - \mu_0 + \mu_1) = 8(\mu_1 + 1)(\mu_0 - 1)(\mu_0 - \mu_1) \leqslant 0$. Thus (33) is satisfied, and the trinoid has no singularity. □

Irreducible trinoids are classified by theorem 2.6 of [UY00]. They correspond to trinoids with non-integer growth ends. Theorem 2.6 of [UY00] states that there exists a trinoid $\mathcal{T}_{\mu_0,\mu_1,\mu_\infty}$ (without assuming that it has a symmetry plane) if and only if (33) holds and

$$\cos^2(\pi\mu_0) + \cos^2(\pi\mu_1) + \cos^2(\pi\mu_\infty) + 2\cos(\pi\mu_0)\cos(\pi\mu_1)\cos(\pi\mu_\infty) < 1, \tag{34}$$

and in this case this trinoid is unique (in this theorem, the β_j ($j = 1, 2, 3$) correspond to our $\mu_j - 1$ ($j = 0, 1, \infty$), the c_j to our $\frac{1-\mu_j^2}{2}$). Irreducible trinoids are also classified in [BPS02]; it is also proved that (34) is equivalent to (32) (in [BPS02] the Δ_j ($j = 1, 2, 3$) correspond to our $\frac{|[\mu_j]|}{2}$ ($j = 0, 1, \infty$)). Pictures of trinoids can be found in [BPS02] (see also [RUY01], example 4.4).

Proposition 52. *If the asymptotic boundary points of the ends of x° are distinct and (28) holds then x° gives a trinoid by Schwarz reflection.*

Proof. For $j = 1, 2, 3$, the oriented curve $\mathcal{L}_j$ and the mean curvature vector of x° on $\mathcal{L}_j$ induce an orientation of the plane $\mathcal{P}_j$. Denote by p_0, p_1 and p_∞ the asymptotic boundaries of the ends of x° at 0, 1 and ∞ respectively. Then $\mathcal{L}_1$ goes from p_∞ to p_0, $\mathcal{L}_2$ goes from p_0 to p_1, and $\mathcal{L}_3$ goes from p_1 to p_∞. Denote by Q_1, Q_2 and Q_3 the asymptotic boundaries of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. These are great circles in $\bar{\mathbb{C}}$. They are given the orientation induced by $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively. We have $p_0 \in Q_1 \cap Q_2$, $p_1 \in Q_2 \cap Q_3$ and $p_\infty \in Q_3 \cap Q_1$. Moreover, the circles are pairwise tangent at these points, and their orientations at these points are compatible (since the boundary lines have turned of an angle π at each end, because of (28)). Since p_0, p_1 and p_∞ are distinct, this is not possible unless the three circles are equal (see figure 8). Consequently, the planes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are equal, and doing the Schwarz reflection of x° with respect to this plane gives a trinoid. □

We now describe what the immersion x° looks like in the case where $(p, q, r) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(p, q, r) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ or $(p, q, r) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. We first notice that $(iU\delta, iV\delta, iW\delta) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(iU\delta, iV\delta, iW\delta) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ and $(iU\delta, iV\delta, iW\delta) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ if and only if $1-\alpha^2-\beta^2+\gamma^2 = 0$, $1 - \alpha^2 + \beta^2 - \gamma^2 = 0$ and $1 + \alpha^2 - \beta^2 - \gamma^2 = 0$ respectively. Henceforth we assume that none of these conditions are satisfied.

If $(p, q, r) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, then by lemma 38 we have $\Lambda(0) \neq 0$, $\Lambda(1) = 0$ and $\Lambda(1) - \Lambda(0) \neq 0$, so $S_z g^\circ - 2Q^\circ$ has poles of order 1 at 0 and ∞ and is holomorphic at 1. This means that the end at 1 is embedded, but the ends at 0 and ∞ are not: the planes $\mathcal{P}_2$ and $\mathcal{P}_3$ are identical, and the plane $\mathcal{P}_1$ is tangent at infinity to $\mathcal{P}_2$ but different. Applying Schwarz reflection infinitely many times, we get a surface that is invariant by the parabolic isometry generated by the

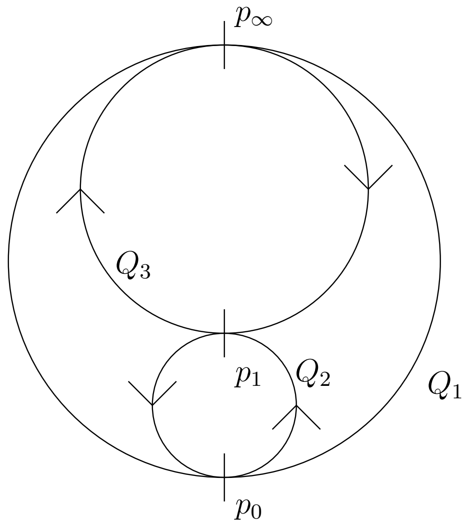

FIGURE 8. The asymptotic boundaries of the boundary planes; there is no orientation of Q_1 compatible with those of Q_2 and Q_3.

reflections with respect to $\mathcal{P}_1$ and $\mathcal{P}_2$. The ends at 0 and ∞ are not annular ends since x° is not single-valued at 0 and ∞.

If $(p,q,r) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, then we have $\Lambda(0) \neq 0$, $\Lambda(1) \neq 0$ and $\Lambda(1) - \Lambda(0) = 0$, so similarly the end at ∞ is embedded and the ends at 0 and 1 are not (and they are not annular ends).

If $(p,q,r) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then we have $\Lambda(0) = 0$, $\Lambda(1) \neq 0$ and $\Lambda(1) - \Lambda(0) \neq 0$, so the end at 0 is embedded and the ends at 1 and ∞ are not (and they are not annular ends).

Then by propostion 52 the asymptotic boundary points of these immersions are not pairwise distinct (otherwise these immersions would give trinoids by Schwarz reflection).

References

[Bal03] A. Balser. On trinoids and minimal disks bounded by lines. Master's thesis, Technische Universität Darmstadt, 2003.

[BPS02] A. Bobenko, T. Pavlyukevich, and B. Springborn. Hyperbolic constant mean curvature one surfaces: Spinor representation and trinoids in hypergeometric functions. arXiv:math.DG/0206021, to appear in Math. Z., 2002.

[Bry87] R. Bryant. Surfaces of mean curvature one in hyperbolic space. *Astérisque*, 154-155:321–347, 1987.

[CHR01] P. Collin, L. Hauswirth, and H. Rosenberg. The geometry of finite topology bryant surfaces. *Ann. of Math. (2)*, 153:1–37, 2001.

[Dan02] B. Daniel. Flux for bryant surfaces and applications to embedded ends of finite total curvature. arXiv:math.DG/0303307, to appear in Illinois J. Math., 2002.

[Kar01] H. Karcher. Hyperbolic constant mean curvature one surfaces with compact fundamental domains. Preprint, 2001.

[KS96] R. Kusner and N. Schmitt. The spinor representation of surfaces in space. arXiv:dg-ga/96100050, 1996.

[MOS66] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*. Springer-Verlag, 1966.

[Rie68] B. Riemann. Les surfaces d'aire minima pour un contour donné. In *Œuvres mathématiques de Riemann (traduites par L. Langel)*, pages 305–352. Librairie scientifique et technique Albert Blanchard, Paris, 1968. Written in 1867 by K. Hattendorf from a manuscript by B. Riemann.

[Ros02] H. Rosenberg. Bryant surfaces. In *The Global Theory of Minimal Surfaces in Flat Spaces, Martina Franca, Italy 1999, Lecture Notes in Mathematics 1775*. Springer, 2002.

[RUY01] W. Rossman, M. Umehara, and K. Yamada. Period problems for mean curvature 1 surfaces in $\mathbb{H}^3$ (with application to surfaces of low total curvature). arXiv:math.DG/0102185, to appear in Advanced Studies in Pure Mathematics, Surveys on Geometry and Integrable Systems, 2001.

[SET01] R. Sá Earp and É. Toubiana. On the geometry of constant mean curvature one surfaces in hyperbolic space. *Illinois J. Math.*, 45(2):371–401, 2001.

[UY93] M. Umehara and K. Yamada. Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space. *Ann. of Math. (2)*, 137:611–638, 1993.

[UY00] M. Umehara and K. Yamada. Metrics of constant curvature 1 with three conical singularities on the 2-sphere. *Illinois J. Math.*, 44(1):72–94, 2000.

[WW63] E. T. Whittaker and G. W. Watson. *A Course of Modern Analysis*. Cambridge University Press, 1963.